A Question of Gross and Weighted Sharing of a Finite Set by Meromorphic Functions *

Indrajit Lahiri†

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Abstract

We prove a uniqueness theorem for meromorphic functions sharing one finite set with weight two and this improves some results of Yi [11], Li and Yang [8] and Fang and Hua [2].

1 Introduction

Let $f$ be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $S \subset \mathbb{C} \cup \{\infty\}$ we define by $E_f(S)$ the set

$$E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where an $a$-point of multiplicity $m$ is counted $m$ times.

In 1976, Gross [3] proved that there exist three finite sets $S_1, S_2, S_3$ such that any two entire functions $f$, $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical. In the same paper Gross asked the following question: Can one find two (or even one) finite sets $S_1$ and $S_2$ such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

A set $S$ for which two meromorphic functions $f$, $g$ satisfying $E_f(S) = E_g(S)$ become identical is called a unique range set of meromorphic functions (cf. [4, 8]).

In 1982, Gross and Yang [4] proved the following theorem.

THEOREM A. Let $S = \{z : e^z + z = 0\}$. If two entire functions $f$, $g$ satisfy $E_f(S) = E_g(S)$ then $f \equiv g$.

Since the set $S = \{z : e^z + z = 0\}$ contains infinitely many elements, the above result does not answer the question of Gross.

In 1994, Yi [10] exhibited a finite set $S$ containing 15 elements which is a unique range set of entire functions and provided an affirmative answer to the question of Gross.

In 1995, Yi [11] and Li and Yang [8] independently proved the following result which gives a better answer to the question of Gross.

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†Department of Mathematics, University of Kalyani, West Bengal 741235, India.
THEOREM B. Let \( S = \{ z : z^7 - z^6 - 1 = 0 \} \). If two entire functions \( f, g \) satisfy \( E_f(S) = E_g(S) \) then \( f \equiv g \).

Extending Theorem B to meromorphic functions, recently Fang and Hua [2] proved the following theorem.

THEOREM C. Let \( S = \{ z : z^7 - z^6 - 1 = 0 \} \). If two meromorphic functions \( f, g \) are such that \( \Theta(\infty; f) > 11/12, \Theta(\infty; g) > 11/12 \) and \( E_f(S) = E_g(S) \) then \( f \equiv g \).

Here \( \Theta \) is the ramification index which is defined below.

In [6, 7] the notion of weighted sharing is introduced which we explain in the following definition.

**DEFINITION 1.** Let \( k \) be a nonnegative integer or infinity. For \( a \in C \cup \{ \infty \} \), we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \), and \( k+1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f \) and \( g \) share the value \( a \) with weight \( k \).

The definition implies that if \( f \), \( g \) share a value \( a \) with weight \( k \) then \( z_0 \) is a zero of \( f-a \) with multiplicity \( m( \leq k ) \) if and only if it is a zero of \( g-a \) with multiplicity \( m( \leq k ) \), and \( z_0 \) is a zero of \( f-a \) with multiplicity \( m( > k ) \) if and only if it is a zero of \( g-a \) with multiplicity \( n( > k ) \) where \( m \) is not necessarily equal to \( n \).

We say that \( f \), \( g \) share \( (a, k) \) if \( f \), \( g \) share the value \( a \) with weight \( k \). Clearly if \( f \), \( g \) share \( (a, k) \) then \( f \), \( g \) share \( (a, p) \) for all integer \( p \) which satisfies \( 0 \leq p < k \). Also we note that \( f \), \( g \) share a value \( a \) IM (ignoring multiplicity) or CM (counting multiplicity) if and only if \( f \), \( g \) share \((a, 0)\) or \((a, \infty)\) respectively.

**DEFINITION 2.** For \( S \subset C \cup \{ \infty \} \), we define \( E_f(S, k) \) as \( E_f(S, k) = \bigcup_{a \in S} E_k(a; f) \), where \( k \) is a nonnegative integer or infinity.

The above definition is in [6]. Clearly \( E_f(S) = E_f(S, \infty) \).

**DEFINITION 3.** A set \( S \) for which two meromorphic functions \( f, g \) satisfying \( E_f(S, k) = E_g(S, k) \) becomes identical is called a unique range set of weight \( k \) for meromorphic functions.

Unless stated otherwise, throughout the paper \( f \) and \( g \) are two nonconstant meromorphic functions. We now explain some basic definitions and notations of the value distribution theory (see e.g. [5]). We denote by \( n(r, f) \) the number of poles of \( f \) in \( |z| \leq r \), where a pole is counted according to its multiplicity, and by \( \pi(r, f) \) the number of distinct poles of \( f \) in \( |z| \leq r \). Also we put

\[
N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,
\]

and

\[
\overline{N}(r, f) = \int_0^r \frac{\pi(t, f) - \pi(0, f)}{t} dt + \pi(0, f) \log r.
\]

The quantities \( N(r, f), \overline{N}(r, f) \) are called respectively the counting function and reduced counting function of poles of \( f \). Let

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,
\]
where $\log^+ x = \log x$ if $x \geq 1$ and $\log^+ x = 0$ if $0 \leq x < 1$. We call $m(r, f)$ the proximity function of $f$. The sum $T(r, f) = m(r, f) + N(r, f)$ is called the Nevanlinna characteristic function of $f$. If $a$ is a finite complex number, we put

$$m(r, a; f) = m \left(r, \frac{1}{f-a}\right), \quad N(r, a; f) = N \left(r, \frac{1}{f-a}\right), \quad \overline{N}(r, a; f) = \overline{N} \left(r, \frac{1}{f-a}\right).$$

The quantity

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}$$

is called the ramification index, where $a \in \mathbb{C} \cup \{\infty\}$ and $\overline{N}(r, \infty; f) = \overline{N}(r, f)$. By the second fundamental theorem we know that the set $\{a : a \in \mathbb{C} \cup \{\infty\}, \Theta(a; f) > 0\}$ is countable and $\sum_a \Theta(a; f) \leq 2$. Finally we denote by $N_2(r, a; f)$ the counting function of $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq 2$ and is counted twice if $m > 2$ (see e.g. [1]).

In this paper we prove the following theorem which improves Theorem B and Theorem C.

**THEOREM 1.** Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If $f$ and $g$ satisfy $\Theta(\infty; f) + \Theta(\infty; g) > 3/2$ and $E_f(S, 2) = E_g(S, 2)$, then $f \equiv g$.

## 2 Preparatory Lemmas

In this section we present some lemmas which will be required to prove our main Theorem. The first one is in [9].

**LEMMA 1.** Let $P(f) = \sum_{j=0}^n a_j f^j$, where $a_0, a_1, \ldots, a_n (\neq 0)$ are such that $T(r, a_j) = S(r, f)$ for $j = 0, 1, \ldots, n$. Then $T(r, P(f)) = nT(r, f) + S(r, f)$.

**LEMMA 2.** If $\Theta(\infty; f) + \Theta(\infty; g) > 3/2$, then for $n \geq 3$, $f^{n-1}(f-1)g^{n-1}(g-1) \equiv 1$.

**PROOF.** Assume to the contrary that

$$f^{n-1}(f-1)g^{n-1}(g-1) \equiv 1. \tag{1}$$

Suppose $f$ does not have any pole. Then from (1) it follows that $g$ has no zero nor 1-point. So by the deficiency relation we get $\Theta(\infty; g) = 0$, which contradicts the given condition. So the lemma is proved in this case. Similarly we can prove the lemma when $g$ does not have any pole. Now we suppose that $f$ and $g$ have poles. From (1), we see that if $z_0$ is a zero of $f$ with multiplicity $p$ then $z_0$ is a pole of $g$ with multiplicity $q$ such that $p(n-1) = nq$, i.e., $p = qn/(n-1)$. Since $n, p, q$ are all positive integers, it follows that $p \geq n$. Hence $\Theta(0; f) \geq 1 - 1/n$. Again from (1), we see that if $z_0$ is an 1-point of $f$ with multiplicity $p$ then $z_0$ is a pole of $g$ with multiplicity $q$ such that $p = qn$ and so $p \geq n$. Hence $\Theta(0; f) \geq 1 - 1/n$. Similarly we can prove that $\Theta(0; g) \geq 1 - 1/n$ and $\Theta(1; g) \geq 1 - 1/n$. So by the deficiency relation we get

$$\Theta(0; f) + \Theta(1; f) + \Theta(0; g) + \Theta(1; g) + \Theta(\infty; f) + \Theta(\infty; g) \leq 4,$$

or,

$$4(1 - \frac{1}{n}) + \frac{3}{2} \leq 4.$$
or $n \leq 8/3$, a contradiction. This proves the lemma.

**Lemma 3.** If $\Theta(\infty; f) + \Theta(\infty; g) > 3/2$, then for $n \geq 4$, $f^{n-1}(f-1) \equiv g^{n-1}(g-1)$ implies $f \equiv g$.

**Proof.** Let

$$f^{n-1}(f-1) \equiv g^{n-1}(g-1).$$

(2)

Assume to the contrary that $f \not\equiv g$. Then from (2) we get

$$f \equiv 1 - \frac{y^{n-1}}{1 + y + y^2 + \cdots + y^{n-1}},$$

(3)

where $y = g/f$. If $y$ is constant then $y \neq 1$. Also from (2) we see that $y^n \neq 1$ and $y^{n-1} \neq 1$ and so (2) implies

$$f \equiv \frac{1 - y^{n-1}}{1 - y^n},$$

which is a contradiction because $f$ is nonconstant. Let $y$ be nonconstant. From (3) we get by the first fundamental theorem and Lemma 1 that

$$T(r, f) = T(r, \sum_{i=0}^{n-1} \frac{1}{y^i}) + S(r, y) = (n-1)T(r, \frac{1}{y}) + S(r, y)$$

$$= (n-1)T(r, y) + S(r, y).$$

Now we note that any pole of $y$ is not a pole of $1 - y^{n-1}/\sum_{j=1}^{n-1} y^j$. So from (3) it follows that

$$\sum_{k=1}^{n-1} N(r, u_k; y) \leq N(r, \infty; f),$$

where $u_k = \exp(2k\pi i/n)$ for $k = 1, 2, \ldots, n-1$. By the second fundamental theorem we get

$$(n-3)T(r, y) \leq \sum_{k=1}^{n-1} N(r, u_k; y) + S(r, y)$$

$$\leq N(r, \infty; f) + S(r, y)$$

$$< (1 - \Theta(\infty; f) + \varepsilon)T(r, f) + S(r, y)$$

$$= (n-1)(1 - \Theta(\infty; f) + \varepsilon)T(r, y) + S(r, y),$$

(4)

where $\varepsilon > 0$.

Again putting $y_1 = 1/y$, noting that $T(r, y) = T(r, y_1) + O(1)$ and proceeding as above we get

$$(n-3)T(r, y) \leq (n-1)(1 - \Theta(\infty; g) + \varepsilon)T(r, y) + S(r, y),$$

(5)

where $\varepsilon > 0$. From (4) and (5) we get in view of the given condition,

$$2(n-3)T(r, y)$$

$$\leq (n-1)(2 - \Theta(\infty; f) - \Theta(\infty; g) + 2\varepsilon)T(r, y) + S(r, y)$$

$$< (n-1)(1 + 2\varepsilon)T(r, y) + S(r, y),$$
which implies a contradiction for all sufficiently small positive \( \varepsilon \) due to the assumption that \( n \geq 4 \). Hence \( f \equiv g \). This completes the proof.

**LEMMA 4.** If \( f, g \) share \((1, 2)\), then one of the following holds: (i) \( T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g) \), where \( T(r) = \max\{T(r, f), T(r, g)\} \), (ii) \( fg \equiv 1 \), or, (iii) \( f \equiv g \).

The proof can be found in [7].

### 3 Proof of Theorem

Let \( F = f^6(f - 1) \) and \( G = g^6(g - 1) \). Since \( E_f(S, 2) = E_f(S, 2) \), it follows that \( F, G \) share \((1, 2)\). Also by Lemma 1, \( T(r, F) = 7T(r, f) + S(r, f) \) and \( T(r, G) = 7T(r, g) + S(r, g) \). Now

\[
N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; G) + N_2(r, \infty; G) + S(r, F) + S(r, G)
\]

\[
\leq 2N(r, 0; f) + N_2(r, 0; f - 1) + 2N(r, 0; g)
\]

\[
+ N_2(r, 0; g - 1) + 2N(r, \infty; f) + 2N(r, \infty; g) + S(r, f) + S(r, g)
\]

\[
\leq \{6 + 2(2 - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon)\} T(r) + S(r, f) + S(r, g)
\]

\[
= (10 - 2\Theta(\infty; f) - 2\Theta(\infty; g) + 2\varepsilon) T(r) + S(r, f) + S(r, g),
\]

where \( \varepsilon > 0 \). Also we see that

\[
\max\{T(r, F), T(r, G)\} = 7T(r) + S(r, f) + S(r, g).
\]

From (6) and (7), we see that

\[
\max\{T(r, F), T(r, G)\} \\
\leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r, F) + S(r, G)
\]

if

\[
7T(r) \leq (10 - 2\Theta(\infty; f) - 2\Theta(\infty; g) + 2\varepsilon) T(r) + S(r, f) + S(r, g)
\]

i.e., if

\[
(2\Theta(\infty; f) + 2\Theta(\infty; g) - 3 - 2\varepsilon) T(r) \leq S(r, f) + S(r, g).
\]

Then a contradiction is reached for sufficiently small positive \( \varepsilon \) because \( \Theta(\infty; f) + \Theta(\infty; g) > 3/2 \). By Lemma 2, we see that \( FG \neq 1 \) because \( \Theta(\infty; f) + \Theta(\infty; g) > 3/2 \). Hence applying Lemma 4, we see that \( F \equiv G \) and so by Lemma 3, we get \( f \equiv g \). This completes the proof.

### References


