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Starlike harmonic functions in parabolic region associated with a convolution structure

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Abstract. Making use of a convolution structure, we introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. The results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for this generalized class of functions

1 Introduction and preliminaries

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

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which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = f'(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \ (0 \le b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (0 \le b_1 < 1). \tag{2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well-known class S of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

For functions $f \in \mathcal{H}$ given by (1) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}, \tag{3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n} \ \ (z \in \mathcal{U}). \tag{4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (4), we consider the following examples.

(1) If

$$F(z) = z + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)\ z^n + \sum_{n=1}^{\infty} \sigma_n(\alpha_1)\ \overline{z}^n \tag{5}$$

and $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1) = \frac{\Theta\Gamma(\alpha_1 + A_1(n-1))\ldots\Gamma(\alpha_p + A_p(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1))\ldots\Gamma(\beta_q + B_q(n-1))}\ , \tag{6}$$

where Θ is given by

$$\Theta = \left(\prod_{m=0}^{p} \Gamma(\alpha_m)\right)^{-1} \left(\prod_{m=0}^{q} \Gamma(\beta_m)\right) \tag{7}$$

and then the convolution(4) gives the Wright's generalized hypergeometric function (see [17])

$${}_p\Psi_q[(\alpha_1, A_1), \ldots; (\beta_1, B_1), \ldots; z] =_p \Psi_q[(\alpha_n, A_n)_{1,p} (\beta_n, B_n)_{1,q}; z]$$

defined by

$${}_p\Psi_q[(\alpha_n, A_n)_{1,p} (\beta_n, B_n)_{1,q}; z] = \sum_{n=0}^{\infty} \{\prod_{m=1}^{p} \Gamma(\alpha_m + nA_m\}\{\prod_{m=1}^{q} \Gamma(\beta_m + nB_m\}^{-1} \frac{z^m}{n!}$$

which was initially studied by Murugusundaramoorthy and Vijaya (see [10]).

(2) If $A_m = 1$ $(m = 1, \ldots, p)$ and $B_m = 1$ $(m = 1, \ldots, q)$, then we have the following relationship

$$F(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n + \sum_{n=1}^{\infty} \Gamma_n \bar{z}^n, \tag{8}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_q)_{n-1}} \frac{1}{(n-1)!},$$

and the convolution (4) gives the Dziok–Srivastava operator (see [5]):

$$\Lambda(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) f(z) \equiv \mathcal{H}_q^p(\alpha_1, \beta_1) f(z),$$

where $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ are positive real numbers, $p \le q+1; p, q \in \mathbb{N} \cup \{0\}$, and $(\alpha)_n$ denotes the familiar Pochhammer symbol (or shifted factorial).

Remark 1 *When* $p = 1$, $q = 1$; $\alpha_1 = a$, $\alpha_2 = 1$; $\beta_1 = c$, *then (8) corresponds to the operator due to Carlson-Shaffer (see [2]) given by*

$$\mathcal{L}(a, c) f(z) := (f * F)(z),$$

where

$$F(z) := z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n + \sum_{n=1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \bar{z}^n \quad (c \ne 0, -1, -2, \ldots). \tag{9}$$

Remark 2 *When* $p = 1$, $q = 0$; $\alpha_1 = k+1$ $(k > -1)$, $\alpha_2 = 1$; $\beta_1 = 1$, *then (8) yields the Ruscheweyh derivative operator (see [8]) given by* $D^k f(z) := (f*F)(z)$ *where*

$$F(z) = z + \sum_{n=2}^{\infty} \binom{k+n-1}{n-1} z^n + \sum_{n=1}^{\infty} \binom{k+n-1}{n-1} \overline{z}^n, \tag{10}$$

which was initially studied by Jahangiri et al. (see [8]).

(3) If $\mathcal{D}^l f(z) = f * F$ where

$$F(z) = z + \sum_{n=2}^{\infty} n^l z^n + (-1)^l \sum_{n=1}^{\infty} n^l \overline{z}^n \quad (\, l \geq 0) \,, \tag{11}$$

was initially studied by Jahangiri et al. (see [9]).

(4) Lastly, if $\mathcal{S}_\alpha f(z) = f * F$ we have

$$F(z) = z + \sum_{m=2}^{\infty} |C_n(\alpha)| z^n + \sum_{n=1}^{\infty} |C_n(\alpha)| \overline{z}^n, \tag{12}$$

and

$$C_n(\alpha) = \frac{\prod_{j=2}^{n}(j - 2\alpha)}{(n-1)!} \quad (n \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \dots\}) \tag{13}$$

which is decreasing in α and satisfies

$$\lim_{n\to\infty} C_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2} \\ 1 & \text{if } \alpha = \frac{1}{2} \\ 0 & \text{if } \alpha > \frac{1}{2} \end{cases} . \tag{14}$$

For the purpose of this paper, we introduce here a subclass of $\mathcal{H}$ denoted by $\mathcal{R}_H(F; \lambda, \gamma)$ which involves the convolution (3) and consist of all functions of the form (1) satisfying the inequality:

$$\text{Re} \left\{ (1 + e^{i\psi}) \frac{z(f(z) * F(z))'}{(1-\lambda)z + \lambda(f(z) * F(z))} - e^{i\psi} \right\} \geq \gamma. \tag{15}$$

Equivalently

$$\text{Re} \left\{ (1 + e^{i\psi}) \frac{z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}}{(1-\lambda)z + \lambda[h(z) * H(z) + \overline{g(z) * G(z)}]} - e^{i\psi} \right\} \geq \gamma \tag{16}$$

where $z \in \mathcal{U},\ 0 \le \lambda \le 1$.

Also denote $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma) = \mathcal{R}_{H}(F;\lambda,\gamma) \bigcap \mathcal{T}_{\mathcal{H}}$ where $\mathcal{T}_{\mathcal{H}}$ is the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}\ \ (0 \le b_1 < 1). \tag{17}$$

called the class of harmonic functions with negative coefficients (see [14]).

It is of special interest to note that for suitable choices of $\lambda = 0$ and $\lambda = 1$ the classes USD [13] and S_p [11] to include the following harmonic functions

$$\text{Re}\ \left\{(1 + e^{i\psi})(f(z) * F(z))' - e^{i\psi}\right\} \ge \gamma,$$

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z(f(z) * F(z))'}{(f(z) * F(z))} - e^{i\psi}\right\} \ge \gamma.$$

We mention below some of the function classes which emerge from the function class $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$ defined above. Indeed, we observe that if we specialize the function F by (5) to (11), and denote the corresponding reducible classes of functions of $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$, respectively, by $\mathcal{W}_q^p(\lambda,\gamma)$, $\mathcal{G}_q^p(\lambda,\gamma)$ $\mathcal{L}_c^a(\lambda,\gamma)$, $\mathcal{R}(k,\lambda,\gamma)$, $\Omega(\lambda,\gamma)$ and $\mathcal{S}(l,\lambda,\gamma)$.

It is of special interest because for suitable choices of F from (15) we can define the following subclasses:
(i) If F is given by (5) we have $(f * F)(z) = W_q^p[\alpha_1]f(z)$ hence we define a class $\mathcal{W}_q^p(\lambda,\gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z(W_q^p[\alpha_1]f(z))'}{(1-\lambda)z + \lambda W_q^p[\alpha_1]f(z)} - e^{i\psi}\right\} \ge \gamma$$

where $W_q^p[\alpha_1]$ is the Wright's generalized operator on harmonic functions (see [10]) .
(ii) If F is given by (8) we have $(f * F)(z) = H_q^p[\alpha_1]f(z)$ hence we define a class $\mathcal{G}_q^p(\lambda,\gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z(H_q^p[\alpha_1]f(z))'}{(1-\lambda)z + \lambda H_q^p[\alpha_1]f(z)} - e^{i\psi}\right\} \ge \gamma$$

where $H_q^p[\alpha_1]$ is the Dziok - Srivastava operator (see [5]).
(iii) $H_1^2([a,1;c]) = \mathcal{L}(a,c)f(z)$, hence we define a class $\mathcal{L}_c^a(\lambda,\gamma)$satisfying the criteria

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z\mathcal{L}(a,c)f(z))'}{(1-\lambda)z + \lambda\mathcal{L}(a,c)f(z)} - e^{i\psi}\right\} \ge \gamma$$

where $\mathcal{L}(a, c)$ is the Carlson - Shaffer operator (see [2]).
(iv) $H_1^2([k+1, 1; 1]) = D^k f(z)$, hence we define a class $\mathcal{R}(k, \lambda, \gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1+e^{i\psi})\frac{z(D^k f(z))'}{(1-\lambda)z+\lambda D^k f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $D^k f(z)(k > -1)$ is the Ruscheweyh derivative operator (see [12]) (also see [8]).
(v) $H_1^2([2, 1; 2-\mu]) = \Omega_z^{\mu} f(z)$ we define another class $\Omega(\lambda, \gamma)$ satisfying the condition

$$\text{Re}\ \left\{(1+e^{i\psi})\frac{z(\Omega_z^{\mu} f(z))'}{(1-\lambda)z+\lambda \Omega_z^{\mu} f(z)} - e^{i\psi}\right\} \geq \gamma$$

given by

$$\Omega_z^{\mu} f(z) = \Gamma(2-\mu) z^{\mu} D_z^{\mu} f(z); (0 \leq \mu < 1)\ ,$$

where Ω_z^{μ} is the Srivastava-Owa fractional derivative operator (see [15]).
(vi) If F is given by (12), we have $S_{\alpha}(z) * f(z) = (f * F)(z)$, hence we define a class $\mathcal{PG}_{\mathcal{H}}(\alpha, \gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1+e^{i\psi})\frac{z(S_{\alpha}(z) * f(z))'}{(1-\lambda)z+\lambda(S_{\alpha}(z) * f(z))} - e^{i\psi}\right\} \geq \gamma, \tag{18}$$

this class was introduced and studied by Vijaya [16] for $\lambda = 1$.
(vii) If F is given by (11), we have $D^l f(z) = (f * F)(z)$, hence we define a class $S(l, \lambda, \gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1+e^{i\psi})\frac{z(D^l f(z))'}{(1-\lambda)z+\lambda D^l f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $D^l f(z); (l \in \mathbb{N})$ is the Sălăgean derivative operator for harmonic functions (see [9]) $\lambda = 1$.

Motivated by the earlier works of (see [6, 9, 17]) on the subject of harmonic functions, in this paper we obtain a sufficient coefficient condition for functions f given by (2) to be in the class $\mathcal{S}_{\mathcal{H}}(F; \lambda, \gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $\mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$. Further, distortion results and extreme points for functions in $\mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$ are also obtained.

For the sake of brevity we denote the corresponding coefficient of F as C_n throughout our study unless otherwise stated.

2 Coefficient bounds

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 1 *Let* $f = h + \overline{g}$ *be given by (2). If*

$$\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n| + \frac{2n+(1+\gamma)\lambda}{1-\gamma}|b_n|\right] C_n \tag{19}$$

where $a_1 = 1$ *and* $0 \le \gamma < 1$, *then* $f \in \mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$.

Proof. We first show that if (19) holds for the coefficients of $f = h + \overline{g}$, the required condition (19) is satisfied. From (16) we can write

$$\begin{aligned}
& \mathrm{Re}\left\{(1+e^{i\psi})\frac{z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}}{(1-\lambda)z + \lambda[h(z)*H(z) + \overline{g(z)*G(z)}]} - e^{i\psi}\right\} \ge \gamma \\
= \; & \mathrm{Re}\left\{\frac{(1+e^{i\psi})[z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}]}{(1-\lambda)z + \lambda[h(z)*H(z) + \overline{g(z)*G(z)}]} - \right. \\
& \left. - \frac{e^{i\psi}[(1-\lambda)z + \lambda(h(z)*H(z) + \overline{g(z)*G(z)})]}{(1-\lambda)z + \lambda[h(z)*H(z) + \overline{g(z)*G(z)}]}\right\} = \\
= \; & \mathrm{Re}\,\frac{A(z)}{B(z)} \ge \gamma
\end{aligned}$$

where

$$\begin{aligned}
A(z) &= (1+e^{i\psi})[z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}] - \\
&\quad - e^{i\psi}[(1-\lambda)z + \lambda(h(z)*H(z) + \overline{g(z)*G(z)})] = \\
&= z + \sum_{n=2}^{\infty}[n + (n-\lambda)e^{i\psi}]C_n a_n z^n - \sum_{n=1}^{\infty}[n + (n-\lambda)e^{i\psi}]C_n \overline{b}_n \overline{z}^n \\
\text{and } B(z) &= (1-\lambda)z + \lambda[h(z)*H(z) + \overline{g(z)*G(z)}] \\
&= z + \sum_{n=2}^{\infty}\lambda C_n a_n z^n + \sum_{n=1}^{\infty}\lambda C_n \overline{b}_n \overline{z}^n.
\end{aligned}$$

Using the fact that Re $\{w\} \ge \gamma$ if and only if $|1-\gamma+w| \ge |1+\gamma-w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \ge 0. \tag{20}$$

Substituting for $A(z)$ and $B(z)$ in (20), we get

$$\begin{aligned}
&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|- \\
= &\left|(2-\gamma)z+\sum_{n=2}^{\infty}[n+(n-\lambda)e^{i\psi}+(1-\gamma)\lambda]C_n a_n z^n-\right. \\
&\left.-\sum_{n=1}^{\infty}[n+(n-\lambda)e^{i\psi}-(1-\gamma)\lambda]C_n\overline{b}_n\ \overline{z}^n\right|- \\
&-\left|-\gamma z+\sum_{n=2}^{\infty}[n+(n-\lambda)e^{i\psi}-(1+\gamma)\lambda]C_n a_n z^n-\right. \\
&\left.-\sum_{n=1}^{\infty}[n+(n-\lambda)e^{i\psi}+(1+\gamma)\lambda]C_n\overline{b}_n\overline{z}^n\right|\geq \\
\geq &(2-\gamma)|z|-\sum_{n=2}^{\infty}[n+(n-\lambda)+(1-\gamma)\lambda]C_n|a_n||z|^n- \\
&-\sum_{n=1}^{\infty}[n+(n-\lambda)-(1-\gamma)\lambda]C_n|b_n|\ |z|^n- \\
&-\gamma|z|-\sum_{n=2}^{\infty}[n+(n-\lambda)-(1+\gamma)\lambda]C_n|a_n|\ |z|^n- \\
&-\sum_{n=1}^{\infty}[n+(n-\lambda)+(1+\gamma)\lambda]C_n|b_n|\ |z|^n\geq \\
\geq &2(1-\gamma)|z|\left\{2-\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n|+\frac{2n+(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n|z|^{n-1}\right\} \\
\geq &2(1-\gamma)\left\{2-\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n|+\frac{2n-(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n\right\}.
\end{aligned}$$

The above expression is non negative by (19), and so $f\in\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$. □

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}x_n z^n+\sum_{n=1}^{\infty}\frac{1-\gamma}{[2n+(1+\gamma)\lambda]C_n}\overline{y}_n(\overline{z})^n \quad (21)$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$ shows that the coefficient bound given by (19) is sharp.

The functions of the form (21) are in $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$ because

$$\sum_{n=1}^{\infty}\left(\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|a_n|+\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|b_n|\right)=$$

$$=1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2.$$

Next theorem establishes that such coefficient bounds cannot be improved further.

Theorem 2 *For* $a_1=1$ *and* $0\leq\gamma<1$, $f=h+\overline{g}\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n|+\frac{2n+(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n\leq 2. \tag{22}$$

Proof. Since $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)\subset\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (17), we notice that the condition

$$\mathrm{Re}\left\{(1+e^{i\psi})\frac{z(h(z)*H(z))'-\overline{z(g(z)*G(z))'}}{(1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}]}-(e^{i\psi}+\gamma)\right\}\geq 0$$

The above inequality is equivalent to

$$\mathrm{Re}\left\{\frac{(1-\gamma)z-\sum\limits_{n=2}^{\infty}[n(1+e^{i\psi})-(1+\gamma+e^{i\psi})\lambda]C_na_nz^n}{z-\sum\limits_{n=2}^{\infty}\lambda C_na_nz^n+\sum\limits_{n=1}^{\infty}\lambda C_n\overline{b}_n\overline{z}^n}-\right.$$

$$\left.-\frac{\sum\limits_{n=1}^{\infty}[n(1+e^{i\psi})+(1+\gamma+e^{i\psi})\lambda]C_n\overline{b}_n\overline{z}^n}{z-\sum\limits_{n=2}^{\infty}\lambda C_na_nz^n+\sum\limits_{n=1}^{\infty}\lambda C_n\overline{b}_n\overline{z}^n}\right\}\geq 0.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, and noting that $\mathrm{Re}(-e^{i\psi})\geq-|e^{i\psi}|=-1$, we must have

$$\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}[2n-(1+\gamma)\lambda]C_na_nr^{n-1}-\sum\limits_{n=1}^{\infty}[2n-(1+\gamma)\lambda]C_nb_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda C_na_nr^{n-1}+\sum\limits_{n=1}^{\infty}\lambda C_nb_nr^{n-1}}\geq 0. \tag{23}$$

If the condition (22) does not hold, then the numerator in (23) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (23) is negative. This contradicts the required condition for $f \in \mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$. This completes the proof of the theorem. □

3 Distortion bounds and extreme points

The following theorem gives the distortion bounds for functions in $\mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$ which yields a covering result for the class $\mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$.

Theorem 3 *Let* $f \in \mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$. *Then for* $|z| = r < 1$, *we have*

$$(1 - b_1)r - \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right)r^2 \le |f(z)|$$
$$\le (1 + b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right)r^2.$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \right| \le (1 + b_1)|z| + \sum_{n=2}^{\infty} (a_n + b_n)|z|^n \le \\
&\le (1 + b_1)r + \sum_{n=2}^{\infty} (a_n + b_n) r^2 \le (1 + b_1)r + \frac{(1-\gamma)}{[4-(1+\gamma)\lambda]C_2} \\
&\quad \sum_{n=2}^{\infty}\left(\frac{[4-(1+\gamma)\lambda]C_2}{(1-\gamma)}a_n + \frac{[4-(1+\gamma)\lambda]C_2}{(1-\gamma)}b_n\right) r^2 \le \\
&\le (1 + b_1)r + \frac{(1-\gamma)1}{[4-(1+\gamma)\lambda]C_2}\left(1 - \frac{1+\gamma}{1-\gamma}b_1\right) r^2 \le \\
&\le (1 + b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right) r^2.
\end{aligned}$$

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. □

The covering result follows from the left hand inequality given in Theorem 3.

Corollary 1 *If* $f(z) \in \mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$, *then*

$$\left\{ w : |w| < \frac{[4-(1+\gamma)\lambda]C_2 - (1-\gamma)}{[4-(1+\gamma)\lambda]C_2} - \frac{[4-(1+\gamma)\lambda]C_2 - (1+\gamma)}{[4-(1+\gamma)\lambda]C_2}|b_1| \right\} \subset f(U).$$

Proof. Using the left hand inequality of Theorem 3 and letting $r \to 1$, we prove that

$$(1-b_1) - \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right) =$$
$$= (1-b_1) - \frac{1}{C_2[4-(1+\gamma)\lambda]}[1-\gamma-(1+\gamma)b_1] =$$
$$= \frac{(1-b_1)C_2[4-(1+\gamma)\lambda] - (1-\gamma) + (1+\gamma)b_1}{C_2[4-(1+\gamma)\lambda]} =$$
$$= \left\{\frac{[4-(1+\gamma)\lambda]C_2 - (1-\gamma)}{[4-(1+\gamma)\lambda]C_2} - \frac{[4-(1+\gamma)\lambda]C_2 - (1+\gamma)}{[4-(1+\gamma)\lambda]C_2}|b_1|\right\} \subset f(U).$$

□

Next we determine the extreme points of closed convex hulls of $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ denoted by $\text{clco}\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 4 *A function* $f(z) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z))$$

where

$$h_1(z) = z, h_n(z) = z - \frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}z^n; \quad (n \geq 2),$$
$$g_n(z) = z + \frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}\bar{z}^n; \quad (n \geq 2),$$
$$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \quad X_n \geq 0 \text{ and } Y_n \geq 0.$$

In particular, the extreme points of $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z)) =$$
$$= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}X_n z^n +$$
$$+ \sum_{n=1}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}Y_n\bar{z}^n$$

Then

$$\sum_{n=2}^{\infty}\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|a_n|+\sum_{n=1}^{\infty}\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|b_n|=$$
$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1,$$

and so $f(z)\in clco\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Conversely, suppose that $f(z)\in clco\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Setting

$$X_n=\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|a_n|,\quad (0\leq X_n\leq 1, n\geq 2)$$

$$Y_n=\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|b_n|,\ \ (0\leq Y_n\leq 1, n\geq 1)$$

and $X_1=1-\sum\limits_{n=2}^{\infty}X_n-\sum\limits_{n=1}^{\infty}Y_n$. Therefore, $f(z)$ can be rewritten as

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n=$$
$$=z-\sum_{n=2}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}X_nz^n+\sum_{n=1}^{\infty}\frac{1-\gamma}{[2n+(1+\gamma)\lambda]C_n}Y_n\overline{z}^n=$$
$$=z+\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n=$$
$$=z\{1-\sum_{n=2}^{\infty}X_n-\sum_{n=1}^{\infty}Y_n\}+\sum_{n=2}^{\infty}h_n(z)X_n+\sum_{n=1}^{\infty}g_n(z)Y_n=$$
$$=\sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))\ \text{ as required.}$$

□

4 Inclusion results

Now we show that $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ is closed under convex combinations of its member and also closed under the convolution product.

Theorem 5 *The family* $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots,$ suppose that $f_i \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \overline{b}_{i,n} \overline{z}^n.$$

Then, by Theorem 2

$$\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{(1-\gamma)} a_{i,n} + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{(1-\gamma)} b_{i,n} \leq 1. \tag{24}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \overline{b}_{i,n} \right) \overline{z}^n.$$

Using the inequality (22), we obtain

$$\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i b_{i,n} \right) =$$
$$= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} a_{i,n} + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} b_{i,n} \right) \leq \sum_{i=1}^{\infty} t_i = 1,$$

and therefore $\sum\limits_{i=1}^{\infty} t_i f_i \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. □

Now, we will examine the closure properties of the class $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt, c > -1.$$

Theorem 6 *Let* $f(z) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. *Then* $\mathcal{L}_c(f(z)) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$

Proof. From the representation of $\mathcal{L}_c(f(z))$, it follows that

$$\begin{aligned}
\mathcal{L}_c(f) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt = \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{n=2}^{\infty}a_nt^n\right)dt+\overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}b_nt^n\right)dt}\right) = \\
&= z-\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_nz^n+\sum_{n=1}^{\infty}\frac{c+1}{c+n}\ b_nz^n.
\end{aligned}$$

Using the inequality (22), we get

$$\sum_{n=1}^{\infty}\left(\frac{[2n-(1+\gamma)\lambda]}{1-\gamma}(\frac{c+1}{c+n}|a_n|)+\frac{[2n+(1+\gamma)\lambda]}{1-\gamma}(\frac{c+1}{c+n}|b_n|)\right)C_n \leq$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{[2n-(1+\gamma)\lambda]}{1-\gamma}|a_n|+\frac{[2n+(1+\gamma)\lambda]}{1-\gamma}|b_n|\right)C_n \leq$$

$$\leq 2(1-\gamma),\ \text{ since } f(z)\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma).$$

Hence by Theorem 2, $\mathcal{L}_c(f(z))\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. □

Concluding remarks

For suitable choices of $F(z)$, as we pointed out the $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$ contains, various function class defined by linear operators such as the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the Sălăgean operator, the fractional derivative operator, and so on. When $\lambda = 0$ and $\lambda = 1$ the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes[1] and [8, 9, 10] respectively. The details involved in the derivations of such specializations of the results presented in this paper are fairly straight- forward, hence omitted.

References

[1] O. P. Ahuja and J. Jahangiri, Noshiro-type harmonic univalent functions, *Sci. Math. Japonica*, **56** (2002), 1–7.

[2] B. C. Carlson and S. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (2002), 737–745.

[3] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Aci. Fenn. Ser. A. I. Math.*, **9** (1984), 3–25.

[4] J. Dziok and Raina, Families of analytic functions associated with the Wright's generalized hypergeometric function, *Demonstratio Math.*, **37** (2004), 533–542.

[5] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Intergral Transform Spec. Funct.*, **14** (2003), 7–18.

[6] J. M. Jahangiri and H. Silverman, Harmonic univalent functions with varying arguments, *Internat. J. Appl. Math.*, **8** (2002), 267–275.

[7] J. M. Jahangiri, Harmonic functions starlike in the unit disc., *J. Math. Anal. Appl.*, **235** (1999), 470–477.

[8] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, Starlikeness of Rucheweyh type harmonic univalent functions, *J. Indian Acad. Math.*, **26** (2004), 191–200.

[9] J. M. Jahangiri, G. Murugusundaramoorthy and K.Vijaya, Salagean-type harmonic univalent functions, *Southwest J. Pure Appl. Math.*, **2** (2002), 77–82.

[10] G. Murugusundaramoorthy and K. Vijaya, A subclass of harmonic functions associated with Wright hypergeometric functions, *Advanced Stud. Contemp. Math.*, **18** (2009), 87–95.

[11] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189–196.

[12] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115.

[13] T. Rosy, B. A. Stephen, K. G. Subramanian and H. Silverman, Classes of convex functions, *Int. J. Maths. and Maths. Anal.*, **23** (2000), 819–825.

[14] H. Silverman, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.*, **220** (1998), 283–289.

[15] H.M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions, *Nagoya Math. J.*, **106** (1987), 1–28.

[16] K. Vijaya, Certain subclass of harmonic prestarlike functions in the parabolic region, *Acta Universitatis Apulensis*, **18** (2009), 1–9.

[17] E. M. Wright,The asymptotic expansion of the generalized hypergeometric function, *Proc. London. Math. Soc.*, **46** (1946), 389–408.

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