Contact warped product semi-slant submanifolds of $(LCS)_n$-manifolds

Shyamal Kumar Hui
Nikhil Banga Sikshan Mahavidyalaya
Bishnupur, Bankura – 722 122
West Bengal, India
email: shyamal_hui@yahoo.co.in

Mehmet Atceken
Gaziosmanpasa University
Faculty of Arts and Sciences,
Department of Mathematics
Tokat – 60250, Turkey
email: matceken@gop.edu.tr

Abstract. The present paper deals with a study of warped product submanifolds of $(LCS)_n$-manifolds and warped product semi-slant submanifolds of $(LCS)_n$-manifolds. It is shown that there exists no proper warped product submanifolds of $(LCS)_n$-manifolds. However we obtain some results for the existence or non-existence of warped product semi-slant submanifolds of $(LCS)_n$-manifolds.

1 Introduction

The notion of warped product manifolds were introduced by Bishop and O’Neill [3] and later it was studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. The existence or non-existence of warped product manifolds plays some important role in differential geometry as well as physics.

The notion of slant submanifolds in a complex manifold was introduced and studied by Chen [7], which is a natural generalization of both invariant and anti-invariant submanifolds. Chen [7] also found examples of slant submanifolds of complex Euclidean space $\mathbb{C}^2$ and $\mathbb{C}^4$. Then Lotta [9] has defined and
studied of slant immersions of a Riemannian manifold into an almost contact metric manifold and proved some properties of such immersions. Also Cabrero et al. [5, 6] studied slant immersions in Sasakian and K-contact manifolds respectively. Again Gupta et al. [8] studied slant submanifolds of a Kenmotsu manifolds and obtained a necessary and sufficient condition for a 3-dimensional submanifold of a 5-dimensional Kenmotsu manifold to be minimal proper slant submanifold.


Recently Shaikh [15] introduced the notion of Lorentzian concircular structure manifolds (briefly, (LCS)$_n$-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [10] and also by Mihai and Rosca [11]. Then Shaikh and Baishya ([17], [18]) investigated the applications of (LCS)$_n$-manifolds to the general theory of relativity and cosmology. The (LCS)$_n$-manifolds is also studied by Sreenivasa et al. [21], Shaikh [16], Shaikh and Binh [19], Shaikh and Hui [20] and others.

The object of the paper is to study warped product semi-slant submanifolds of (LCS)$_n$-manifolds. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 deals with a study of warped product submanifolds of (LCS)$_n$-manifolds. It is shown that there do not exist proper warped product submanifolds $N = N_1 \times f N_2$ of a (LCS)$_n$-manifold $M$, where $N_1$ and $N_2$ are submanifolds of $M$. In section 4, we investigate warped product semi-slant submanifolds of (LCS)$_n$-manifolds and obtain many interesting results.

## 2 Preliminaries

An $n$-dimensional Lorentzian manifold $M$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $g$, that is, $M$ admits a smooth symmetric tensor field $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature
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\((-,+\cdots,+\)), where \(T_pM\) denotes the tangent vector space of \(M\) at \(p\) and \(\mathbb{R}\) is the real number space. A non-zero vector \(v \in T_pM\) is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies \(g_p(v,v) < 0\) (resp., \(\leq 0\), \(= 0\), \(> 0\)) [12].

**Definition 1** [15] In a Lorentzian manifold \((M, g)\) a vector field \(P\) defined by

\[ g(X, P) = A(X), \]

for any \(X \in \Gamma(TM)\), is said to be a concircular vector field if

\[ (\bar{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\} \]

where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form and \(\bar{\nabla}\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\).

Let \(M\) be an \(n\)-dimensional Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[ g(\xi, \xi) = -1. \] (1)

Since \(\xi\) is a unit concircular vector field, it follows that there exists a non-zero 1-form \(\eta\) such that for

\[ g(X, \xi) = \eta(X), \] (2)

the equation of the following form holds

\[ (\bar{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \] (3)

for all vector fields \(X, Y\), where \(\bar{\nabla}\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfies

\[ \nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho \eta(X), \] (4)

\(\rho\) being a certain scalar function given by \(\rho = -\langle \xi, \alpha \rangle\). Let us take

\[ \phi X = \frac{1}{\alpha} \bar{\nabla}_X \xi, \] (5)

then from (3) and (5) we have

\[ \phi X = X + \eta(X)\xi, \] (6)

from which it follows that \(\phi\) is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold \(M\) together with the
unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and an $(1,1)$ tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [15]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [10]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold [15]:

$$\eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (7)$$

$$\phi^2 X = X + \eta(X)\xi, \quad (8)$$

$$S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \quad (9)$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (10)$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \quad (11)$$

$$(\bar{\nabla}_X \phi)(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X), \quad (12)$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \quad (13)$$

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi, \quad (14)$$

for all $X, Y, Z \in \Gamma(TM)$ and $\beta = -\langle \xi, \rho \rangle$ is a scalar function, where $R$ is the curvature tensor and $S$ is the Ricci tensor of the manifold.

Let $N$ be a submanifold of a $(LCS)_n$-manifold $M$ with induced metric $g$. Also let $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TN$ and the normal bundle $T^\perp N$ of $N$ respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_XY = \nabla_X Y + h(X, Y) \quad (15)$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V \quad (16)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp N)$, where $h$ and $A_V$ are second fundamental form and the shape operator (corresponding to the normal vector field $V$) respectively for the immersion of $N$ into $M$. The second fundamental form $h$ and the shape operator $A_V$ are related by [22]

$$g(h(X, Y), V) = g(A_V X, Y) \quad (17)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp N)$.

For any $X \in \Gamma(TM)$, we may write

$$\phi X = EX + FX, \quad (18)$$
where $EX$ is the tangential component and $FX$ is the normal component of $\phi X$.

Also for any $V \in \Gamma(T^1N)$, we have

$$\phi V = BV + CV,$$

where $BV$ and $CV$ are the tangential and normal components of $\phi V$ respectively. From (18) and (19) we can derive the tensor fields $E, F, B$ and $C$ are also symmetric. The covariant derivatives of the tensor fields of $E$ and $F$ are defined as

$$\nabla_X E(Y) = \nabla_X EY - E(\nabla_X Y),$$

$$\nabla_X F(Y) = \nabla_X FY - F(\nabla_X Y)$$

for all $X, Y \in \Gamma(TN)$. The canonical structures $E$ and $F$ on a submanifold $N$ are said to be parallel if $\nabla E = 0$ and $\bar{\nabla} F = 0$ respectively.

Throughout the paper, we consider $\xi$ to be tangent to $N$. The submanifold $N$ is said to be invariant if $F$ is identically zero, i.e., $\phi X \in \Gamma(TN)$ for any $X \in \Gamma(TN)$. Also $N$ is said to anti-invariant if $E$ is identically zero, that is $\phi X \in \Gamma(T^1N)$ for any $X \in \Gamma(TN)$.

Furthermore for submanifolds tangent to the structure vector field $\xi$, there is another class of submanifolds which is called slant submanifold. For each non-zero vector $X$ tangent to $N$ at $x$, the angle $\theta(x)$, $0 \leq \theta(x) \leq \frac{\pi}{2}$ between $\phi X$ and $EX$ is called the slant angle or Wirtinger angle. If the slant angle is constant, then the submanifold is also called the slant submanifold. Invariant and anti-invariant submanifolds are particular slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. A slant submanifold is said to be proper slant if the slant angle $\theta$ lies strictly between $0$ and $\frac{\pi}{2}$, i.e., $0 < \theta < \frac{\pi}{2}$.

**Lemma 1** [5] Let $N$ be a submanifold of a $(LCS)_n$-manifold $M$ such that $\xi$ is tangent to $N$. Then $N$ is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$E^2 = \lambda(I + \eta \otimes \xi).$$

Furthermore, if $\theta$ is the slant angle of $N$, then $\lambda = \cos^2 \theta$.

Also from (22) we have

$$g(EX, EY) = \cos^2 \theta [g(X, Y) + \eta(X)\eta(Y)],$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) + \eta(X)\eta(Y)]$$

for any $X, Y$ tangent to $N$.

The study of semi-slant submanifolds of almost Hermitian manifolds was introduced by Papaghuic [13], which was extended to almost contact manifold...
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by Cabrerizo et. al \[4\]. The submanifold \(N\) is called semi-slant submanifold of \(M\) if there exist an orthogonal direct decomposition of \(T_N\) as

\[ T_N = D_1 \oplus D_2 \oplus \{\xi\}, \]

where \(D_1\) is an invariant distribution, i.e., \(\phi(D_1) = D_1\) and \(D_2\) is slant with slant angle \(\theta \neq 0\). The orthogonal complement of \(FD_2\) in the normal bundle \(T^\perp N\) is an invariant subbundle of \(T^\perp N\) and is denoted by \(\mu\). Thus we have

\[ T^\perp N = FD_2 \oplus \mu. \]

Similarly \(N\) is called anti-slant subbundle of \(M\) if \(D_1\) is an anti-invariant distribution of \(N\), i.e., \(\phi D_1 \subset T^\perp N\) and \(D_2\) is slant with slant angle \(\theta \neq 0\).

3 Warped product submanifolds of \((\text{LCS})_n\)-manifolds

The notion of warped product manifolds were introduced by Bishop and O’Neill \[3\].

**Definition 2** Let \((N_1, g_1)\) and \((N_2, g_2)\) be two Riemannian manifolds and \(f\) be a positive definite smooth function on \(N_1\). The warped product of \(N_1\) and \(N_2\) is the Riemannian manifold \(N_1 \times_f N_2 = (N_1 \times N_2, g)\), where

\[ g = g_1 + f^2 g_2. \] (25)

A warped product manifold \(N_1 \times_f N_2\) is said to be trivial if the warping function \(f\) is constant.

More explicitly, if the vector fields \(X\) and \(Y\) are tangent to \(N_1 \times_f N_2\) at \((x, y)\) then

\[ g(X, Y) = g_1(\pi_1 \ast X, \pi_1 \ast Y) + f^2(x) g_2(\pi_2 \ast X, \pi_2 \ast Y), \]

where \(\pi_i\) \((i = 1, 2)\) are the canonical projections of \(N_1 \times N_2\) onto \(N_1\) and \(N_2\) respectively and \(\ast\) stands for the derivative map.

Let \(N = N_1 \times_f N_2\) be warped product manifold, which means that \(N_1\) and \(N_2\) are totally geodesic and totally umbilical submanifolds of \(N\) respectively. For warped product manifolds, we have \[3\]

**Proposition 1** Let \(N = N_1 \times_f N_2\) be a warped product manifold. Then

(I) \(\nabla_X Y \in TN_1\) is the lift of \(\nabla_X Y\) on \(N_1\)

(II) \(\nabla_U X = \nabla_X U = (X \ln f) U\)

(III) \(\nabla_U V = \nabla_U' V - g(U, V) \nabla \ln f\)
for any $X$, $Y \in \Gamma(TN_1)$ and $U$, $V \in \Gamma(TN_2)$, where $\nabla$ and $\nabla'$ denote the Levi-Civita connections on $N_1$ and $N_2$ respectively.

We now prove the following:

**Theorem 1** There exist no proper warped product submanifolds in the form $N = N_1 \times_f N_2$ of a (LCS)$_n$-manifold $M$ such that $\xi$ is tangent to $N_T$, where $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds of $M$, respectively.

**Proof.** We suppose that $N = N_1 \times_f N_\perp$ is a warped product submanifold of (LCS)$_n$-manifold $M$. For any $X \in \Gamma(TN_T)$ and $U$, $V \in \Gamma(TN_\perp)$, from Proposition 1 we have

$$\nabla_U X = \nabla_X U = (X \ln f) U. \quad (26)$$

On the other hand, by using (12) and (26) we have

$$\begin{align*}
(X \ln f) g(U, V) &= g(\nabla_U X, V) + g(\nabla_U V, X) = g(\phi \nabla_U X, \phi V) \\
&= g(h(U, \phi X), \phi V) - \alpha \eta(X) g(U, \phi V) \\
&= g(h(U, \phi X), \phi V) = g(\phi \nabla_U \phi X, \phi V) \\
&= g(h(\phi X, V), \phi U) = g(\phi \nabla_X \phi U, \phi V) \\
&= -g(A \phi U, \phi X) = -g(h(\phi X, V), \phi U) = -g(\phi \nabla_X \phi U, \phi V) \\
&= -g(\phi \nabla_X U, \phi U) = -g(\nabla_X U, \phi U) = -(X \ln f) g(U, V).
\end{align*}$$

It follows that $X(\ln f) = 0$. So $f$ is constant on $N_T$. Hence we get our desired assertion.

## 4 Warped product semi-slant submanifolds of (LCS)$_n$-manifolds

Let us suppose that $N = N_1 \times_f N_2$ be a warped product semi-slant submanifold of a (LCS)$_n$-manifold $M$. Such submanifolds are always tangent to the structure vector field $\xi$. If the submanifolds $N_\theta$ and $N_T$ (respectively $N_\perp$) are slant and invariant (respectively anti-invariant) submanifolds of a (LCS)$_n$-manifold $M$, then their warped product semi-slant submanifolds may be given by one of the following forms:

(i) $N_1 \times_f N_\theta$ (ii) $N_\perp \times_f N_\theta$ (iii) $N_\theta \times_f N_T$ (iv) $N_\theta \times_f N_\perp$.

However, the existence or non-existence of a structure on a manifold is very important. Because the every structure of a manifold may not be admit. In
this paper, we have researched cases that there exist no warped product semi-
slant submanifolds in a (LCS)$_n$-manifold. Therefore we now study each of the
above four cases and begin the following Theorem:

**Theorem 2** There exist no proper warped product semi-slant submanifold in
the form $N = N_T \times \mathbf{f} N_\Theta$ of a (LCS)$_n$-manifold $M$ such that $\xi$ is tangent to $N_T$,
where $N_T$ and $N_\Theta$ are invariant and slant submanifolds of $M$, respectively.

**Proof.** Let us assume that $N = N_T \times \mathbf{f} N_\Theta$ is a proper warped product semi-
slant submanifolds of a (LCS)$_n$-manifold $M$ such that $\xi$ is tangent to $N_T$. Then
for any $X, \xi \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\Theta)$, from (5) and (15) we have
\[
\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) = \alpha \phi U. \tag{27}
\]
From the tangent and normal components of (27), respectively, we obtain
\[
\xi(\ln f)U = \alpha EU \quad \text{and} \quad h(U, \xi) = \alpha FU. \tag{28}
\]
On the other hand, by using (7) and (12), we have
\[
(\bar{\nabla}_U \phi) \xi = -\phi \bar{\nabla}_U \xi, \quad \alpha U = \phi(\xi(\ln f)U) + \phi h(U, \xi),
\]
that is,
\[
B(U, \xi) + \xi(\ln f)EU = \alpha U \quad \text{and} \quad \xi(\ln f)FU + Ch(U, \xi) = 0. \tag{29}
\]
Since $\Gamma(\mu)$ and $\Gamma(F(TN_\Theta))$ are orthogonal subspaces, we can derive $\xi(\ln f)FU = 0$. So we conclude $\xi(\ln f) = 0$ or $FU = 0$. Here we have to show that $FU$ for
the proof. For this we assume that $FU \neq 0$.

Making use of (12), (15), (16) and (18), we obtain
\[
(\bar{\nabla}_X \phi)U = \nabla_X \phi U - \phi \nabla_X U
\]
\[
h(X, EU) - A_{FU} X + \nabla_X FU = X(\ln f)FU + Bh(X, U) + Ch(X, U). \tag{30}
\]
Taking into account that the tangent components of (30) and making the
necessary abbreviations, we get
\[
A_{FU} X = -Bh(X, U). \tag{31}
\]
With similar thoughts, we have
\[
(\bar{\nabla}_U \phi)X = \bar{\nabla}_U \phi X - \phi \nabla_U X
\]
\[
\alpha \eta(X)U = EX(\ln f)U + h(U, EX) - X(\ln f)EU - X(\ln f)FU
- Bh(X, U) - Ch(X, U). \tag{32}
\]
From the normal components of (32), we arrive at
\[ X(\ln f)FU = h(U, EX) - Ch(U, X). \] (33)

Thus by using (31) and (33), we conclude
\[ X(\ln f)g(FU, FU) = g(h(U, EX), FU) = g(Bh(EX, U), FU) = -g(h(U, EX), FU). \]

This tells us that \( X(\ln f) = 0 \), that is, \( f \) is a constant function \( N_T \) because \( FU \) is a non-null vector field and \( N_\theta \) is a proper slant submanifold.

**Theorem 3** There exist no proper warped product semi-slant submanifolds in the form \( N = N_\perp \times f N_\theta \) of a \((LCS)_n\)-manifold \( M \) such that \( \xi \) is tangent to \( N \), where \( N_\perp \) and \( N_\theta \) are anti-invariant and proper slant submanifolds of \( M \) respectively.

**Proof.** Let \( N = N_\perp \times f N_\theta \) be a proper warped product semi-slant submanifold of a \((LCS)_n\)-manifold \( M \) such that \( \xi \) is tangent to \( N \). If \( \xi \) is tangent to \( \Gamma(TN_\theta) \), then for any \( X \in \Gamma(TN_\theta) \) and \( U \in \Gamma(TN_\perp) \), from (5) and (15), we have
\[ \bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) = \alpha f U, \] (34)
which is equivalent to \( U(\ln f)\xi = 0 \) because \( \xi \neq 0 \). So \( f \) is a constant function on \( N_\perp \).

On the other hand, if \( \xi \in \Gamma(TN_\perp) \), from (5) and (15), we reach
\[ \bar{\nabla}_X \xi = \alpha f \xi + h(X, \xi), \]
that is,
\[ \alpha EX = \xi(\ln f)X \quad \text{and} \quad \alpha FX = h(X, \xi). \] (35)

Furthermore, since \( \phi \xi = 0 \), by direct calculations, we obtain
\[ (\bar{\nabla}_X \phi)\xi = -\phi(\bar{\nabla}_X \xi) \]
\[ \alpha X = \xi(\ln f)EX + \xi(\ln f)FX + Bh(X, \xi) + Ch(X, \xi). \]

It follows that
\[ \alpha X = \xi(\ln f)EX + Bh(X, \xi) \quad \text{and} \quad \xi(\ln f)FX = -Ch(X, \xi). \] (36)
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By virtue of (36), we conclude

\[ \xi(\ln f)g(FX, FX) = \sin^2 \theta \xi(\ln f)g(X, X) = -g(Ch(X, \xi), FX) = 0, \]

which follows $\xi(\ln f) = 0$ or $\sin^2 \theta g(X, X) = 0$. Here if $\xi(\ln f) \neq 0$ and $\sin^2 \theta g(X, X) = 0$, the proof is obvious. Otherwise, making use of (36), we conclude that

\[ \alpha g(X, X) = g(Bh(X, \xi), X) = 0. \]

Consequently, we can easily to see that $\alpha = 0$. This is a contradiction because the ambient space $M$ is a $(\text{LCS})_n$-manifold. Thus the proof is complete.

**Theorem 4** There exist no proper warped product semi-slant submanifolds in the form $N_\theta \times_f N_T$ in $(\text{LCS})_n$-manifold $M$ such that $\xi$ tangent to $N_T$, where $N_\theta$ and $N_T$ are proper slant and invariant submanifolds of $M$.

**Proof.** Let $N = N_\theta \times_f N_T$ be warped product semi-slant submanifolds in a $(\text{LCS})_n$-manifold $M$ such that $\xi$ is tangent to $N_T$. Then for any $\xi, X \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\theta)$, taking account of relations (12), (15), (16), (18) and (19) and Proposition 1, we have

\[ (\bar{\nabla}_U\phi)X = \bar{\nabla}_U\phi X - \phi \bar{\nabla}_U X \]

\[ \alpha \eta(X)U = h(U, EX) - Bh(U, X) - Ch(U, X), \]

which implies that

\[ \alpha \eta(X)U = -Bh(U, X) \text{ and } h(U, EX) = Ch(U, X). \]  

(37)

In the same way, we have

\[ (\bar{\nabla}_X\phi)U = \bar{\nabla}_X\phi U - \phi \bar{\nabla}_X U \]

\[-A_{FU}X + \nabla^\perp_XFU + h(X, EU) = Bh(X, U) + Ch(X, U), \]

from here

\[ Bh(X, U) = -A_{FU}X + EU(\ln f)X - U(\ln f)EX \]

(38)

and

\[ \nabla^\perp_XFU = Ch(X, U) - h(X, EU). \]

(39)
Taking inner product both of sides of (37) with \( V \in \Gamma(TN) \) and also using (38), we arrive at

\[
\alpha\eta(U, V) = -g(Bh(U, X), V) = -g(h(U, X), \phi V) = -g(h(U, X), FV) = g(Bh(X, V), U).
\]

Here for \( X = \xi \), we obtain \( \alpha g(U, V) = 0 \). Because the ambient space \( M \) is a \((\text{LCS})_n\)-manifold and \( N_\theta \) is a proper slant submanifold, this also tells us the accuracy of the statement of the theorem.

**Theorem 5** There exist no proper warped product semi-slant submanifolds in the form \( N = N_\theta \times_f N_\bot \) in a \((\text{LCS})_n\)-manifold such that \( \xi \) tangent to \( N_\theta \), where \( N_\theta \) and \( N_\bot \) are proper slant and anti-invariant submanifolds of \( M \), respectively.

**Proof.** Let us assume that \( N = N_\theta \times_f N_\bot \) be a proper warped product semi-slant submanifold in the \((\text{LCS})_n\)-manifold \( M \) such that \( \xi \) is tangent to \( N_\theta \). Then for \( X \in \Gamma(TN) \) and \( U \in \Gamma(TN_\bot) \), we have

\[
(\nabla_X \phi)U = \nabla_X \phi U - \phi \nabla_X U
\]

\[
-A_F u + \nabla^\bot_X FU = \phi \nabla_X U + \phi h(U, X),
\]

which follows that

\[
A_F u = Bh(X, U) \quad \text{and} \quad (\nabla_X F)U = Ch(X, U).
\]

(40)

In the same way, we have

\[
(\nabla_U \phi)X = \nabla_U \phi X - \phi \nabla_U X,
\]

which also follow that

\[
\alpha\eta(X)U = EX(ln f)U - A_F u U - Bh(X, U),
\]

(41)

\[
\nabla_U^\bot FX = X(ln f)FU + Ch(X, U) - h(U, EX).
\]

(42)

From (41), we can derive

\[
g(h(U, X), FX) = g(h(U, X), FU) = 0.
\]

(43)

Taking \( X = \xi \), in (42), we have \( \xi(ln f)FU = -Ch(X, \xi) \), that is, \( \xi(ln f)FU = 0 \). Let \( X = \xi \) be in (41), then we get

\[
\alpha U = Bh(U, \xi).
\]

(44)
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Taking the inner product of the both sides of (44) by \(U \in \Gamma(T\mathbb{N}_\perp)\), and using (43) we conclude

\[ \alpha g(U, U) = g(Bh(U, \xi), U) = g(h(U, \xi), FU) = 0, \]

which implies that \(\alpha = 0\). This is impossible because the ambient space is a \((\text{LCS})_n\)-manifold. Hence the proof is complete.

**References**


*Received: March 15, 2011*