Spaces of entire functions represented by vector valued Dirichlet series of slow growth

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Abstract. Spaces of all entire functions $f$ represented by vector valued Dirichlet series and having slow growth have been considered. These are endowed with a certain topology under which they become a Frechet space. On this space the form of linear continuous transformations is characterized. Proper bases have also been characterized in terms of growth parameters.

1 Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n}, \quad s = \sigma + it \quad (\sigma, t \text{ are real variables}),$$

where $\{a_n\}$ is a sequence of complex numbers and the sequence $\{\lambda_n\}$ satisfies the conditions $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n \ldots$, $\lambda_n \to \infty$ as $n \to \infty$ and

$$\lim_{n \to \infty} \sup_n \frac{n}{\lambda_n} = D < \infty,$$

$$\lim_{n \to \infty} \sup_{n} (\lambda_{n+1} - \lambda_n) = h > 0,$$

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and

$$\lim_{n \to \infty} \sup \frac{\log |a_n|}{\lambda_n} = -\infty.$$  (4)

By giving different topologies on the set of entire functions represented by the Dirichlet series, Kamthan and Hussain [2] have studied various properties of this space.

Now let $a_n \in E, n = 1, 2, \ldots$, where $(E, \| \cdot \|)$ is a complex Banach space and (4) is replaced by the condition

$$\lim_{n \to \infty} \sup \frac{\log \|a_n\|}{\lambda_n} = -\infty.$$  (5)

Then the series in (1) is called a vector valued Dirichlet series and represents an entire function $f(s)$. In what follows, the series in (1) will represent a Vector valued entire Dirichlet series.

Let for entire functions defined as above by (1) and satisfying (2), (3) and (5),

$$M(\sigma, f) = M(\sigma) = \sup_{-\infty < t < \infty} ||f(\sigma + it)||.$$

Then $M(\sigma)$ is called the maximum modulus of $f(\sigma)$. The order $\rho$ of $f(\sigma)$ is defined as [1]

$$\rho = \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\sigma}, \quad 0 \leq \rho \leq \infty$$  (6)

Also, for $0 < \rho < \infty$ the type $T$ of $f(s)$ is defined by [1]

$$T = \lim_{\sigma \to \infty} \sup \frac{\log M(\sigma)}{e^{\sigma \rho}}, \quad 0 \leq T \leq \infty.$$

It was proved by Srivastava [1] that if $f(s)$ is of order $\rho$ $(0 < \rho < \infty)$ and (2) holds then $f(s)$ is of type $T$ if and only if

$$T = \lim_{n \to \infty} \sup \frac{\lambda_n^{1/\rho} ||a_n||^{1/\lambda_n}}{\rho e}.$$

This implies

$$\lim_{n \to \infty} \sup \lambda_n^{1/\rho} ||a_n||^{1/\lambda_n} = (T e)^{1/\rho}.$$  (7)

We now denote by $X$ the set of all vector valued entire functions $f(s)$ given by (1) and satisfying (2), (3) and (5) for which

$$\lim_{\sigma \to \infty} \sup \frac{\log M(\sigma)}{e^{\sigma \rho}} \leq T < \infty, \quad 0 < \rho < \infty.$$
Then from (7), we have

$$\lim_{n \to \infty} \sup \lambda_n^{1/\rho} \|a_n\|^{1/\lambda_n} \leq (T \rho e)^{1/\rho}. \quad (8)$$

From (8), for arbitrary $\epsilon > 0$ and all $n > n_0 (\epsilon)$,

$$\|a_n\| \left[ \frac{\lambda_n}{(T + \epsilon \rho e^\rho)} \right]^{\lambda_n/\rho} < 1.$$

Hence, if we put

$$\|f\|_q = \sum_{n \geq 1} \|a_n\| \left[ \frac{\lambda_n}{(T + q^{-1}) \rho e^\rho} \right]^{\lambda_n/\rho} \quad q \geq 1,$$

then $\|f\|_q$ is well defined and for $q_1 \leq q_2$, $\|f\|_{q_1} \leq \|f\|_{q_2}$. This norm induces a metric topology on $X$. We define

$$\lambda(f, g) = \sum_{q \geq 1} \frac{1}{2^q} \cdot \frac{\|f - g\|_q}{1 + \|f - g\|_q}.$$

We denote the space $X$ with the above metric $\lambda$ by $X_\lambda$. Various properties of bases of the space $X_\lambda$ using the growth properties of the entire vector valued Dirichlet series have been obtained in [3]. These results obviously do not hold if the order $\rho$ of the entire function $f(s)$ is zero. In this paper we have introduced a metric on the space of entire function of zero order represented by vector valued Dirichlet series thereby obtaining various properties of this space.

## 2 Main results

The vector valued entire function $f(s)$ represented by (1), for which order $\rho$ defined by (6) is equal to zero, we define the logarithmic order $\rho^*$ by

$$\rho^* = \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\log \sigma}, \quad 1 \leq \rho^* \leq \infty.$$

For $1 < \rho^* < \infty$ the logarithmic type $T^*$ is defined by

$$T^* = \lim_{\sigma \to \infty} \sup \frac{\log M(\sigma)}{\sigma^{\rho^*}}, \quad 0 \leq T^* \leq \infty.$$
In [4] the authors have established that \( f(s) \) is of logarithmic order \( \rho^* \), \( 1 < \rho^* < \infty \), and logarithmic type \( T^* \), \( 0 < T^* < \infty \), if and only if

\[
\lim_{n \to \infty} \sup_{\lambda} \frac{\lambda_n \phi(\lambda_n)}{\log \|a_n\|^{-1}} = \frac{\rho^*}{(\rho^* - 1)}(\rho^* T^*)^{1/(\rho^* - 1)},
\]

(9)

where \( \phi(t) \) is the unique solution of the equation \( t = \sigma^\rho - 1 \). The above formula can be proved on the same lines as for ordinary Dirichlet series in [5]. Let \( Y \) denote the set of all entire functions \( f(s) \) given by (1) and satisfying (2), (3), and (5), for which

\[
\lim_{\sigma \to \infty} \sup \log M(\sigma) = T^* < \infty, \quad 0 < \rho^* < \infty.
\]

Then from (9) we have

\[
\lim_{n \to \infty} \sup \frac{\lambda_n \phi(\lambda_n)}{\log \|a_n\|^{-1}} \leq \frac{\rho^*}{(\rho^* - 1)}(\rho^* T^*)^{1/(\rho^* - 1)},
\]

(10)

where \( \phi(\lambda_n) = \lambda_n^{1/\rho^*-1} \). From (10), for arbitrary \( \epsilon > 0 \) and all \( n > n_0(\epsilon) \),

\[
\|a_n\| \leq \exp \left[ -\frac{\lambda_n \phi(\lambda_n)}{\{K \cdot (T^* + \epsilon)\}^{1/(\rho^* - 1)}} \right],
\]

(11)

where \( K = (\rho^* / (\rho^* - 1))^{(\rho^* - 1)} \) be a constant. For each \( f \in Y \), we define the norm

\[
\|f\|_\alpha = \sum_{n \geq 1} \|a_n\| \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \cdot (T^* + \alpha^{-1})\}^{1/(\rho^* - 1)}} \right], \quad \alpha \geq 1
\]

then \( \|f\|_\alpha \) is well defined and for \( \alpha_1 \leq \alpha_2 \), \( \|f\|_{\alpha_1} \leq \|f\|_{\alpha_2} \). This norm induces a metric topology on \( Y \) defined by

\[
d(f, g) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \cdot \frac{\|f - g\|_\alpha}{1 + \|f - g\|_\alpha}.
\]

We denote the space \( Y \) with the above metric \( d \) by \( Y_d \). Now we prove

**Theorem 1** The space \( Y_d \) is a Fréchet space.
Proof. Here, \( Y_d \) is a normed linear metric space. For showing that \( Y_d \) is a Frechet space, we need to show that \( Y_d \) is complete. Hence, let \( \{f_p\} \) be a Cauchy sequence in \( Y_d \). Therefore, for any given \( \varepsilon > 0 \) there exists an integer \( n_0 = n_0(\varepsilon) \) such that
\[
d(f_p, f_q) < \varepsilon \; \forall \; p, q > n_0.
\]
Hence \( \|f_p - f_q\|_\alpha < \varepsilon \; \forall \; p, q > n_0, \alpha \geq 1 \).

Denoting by \( f_p(s) = \sum_{n=1}^{\infty} a_n^{(p)} e^{s\lambda_n}, f_q(s) = \sum_{n=1}^{\infty} a_n^{(q)} e^{s\lambda_n} \), we have therefore
\[
\sum_{n=1}^{\infty} \|a_n^{(p)} - a_n^{(q)}\| \cdot \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{\{K \cdot p^*(T^* + \alpha^{-1})\}^{1/(\rho^* - 1)}} \right] < \varepsilon \quad (12)
\]
for all \( p, q > n_0, \alpha \geq 1 \). Since \( \lambda_n \to \infty \) as \( n \to \infty \), therefore we have \( \|a_n^{(p)} - a_n^{(q)}\| < \varepsilon \; \forall \; p, q \geq n_0 \), and \( n = 1, 2, \ldots \), i.e. for each fixed \( n = 1, 2, \ldots \), \( \{a_n^{(p)}\} \) is a Cauchy sequence in the Banach space \( E \).

Hence there exists a sequence \( \{a_n\} \subseteq E \) such that
\[
\lim_{n \to \infty} a_n^{(p)} = a_n, \; \forall n \geq 1.
\]
Now letting \( q \to \infty \) in (12), we have for \( p \geq n_0 \),
\[
\sum_{n=1}^{\infty} \|a_n^{(p)} - a_n\| \cdot \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{\{K \cdot p^*(T^* + \alpha^{-1})\}^{1/(\rho^* - 1)}} \right] < \varepsilon \quad (13)
\]
Taking \( p = n_0 \), we get for a fixed \( \alpha \) in (12)
\[
\|a_n\| \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{\{K \cdot p^*(T^* + \alpha^{-1})\}^{1/(\rho^* - 1)}} \right] < \\
\|a_n^{(n_0)}\| \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{\{K \cdot p^*(T^* + \alpha^{-1})\}^{1/(\rho^* - 1)}} \right] + \varepsilon
\]
Now \( f^{(n_0)} = \sum_{n=1}^{\infty} a_n^{(n_0)} e^{s\lambda_n} \in Y_d \), hence the condition (11) is satisfied. For arbitrary \( \alpha < \beta \), we have, \( \|a_n^{(n_0)}\| < \exp \left[ \frac{-\lambda_n \Phi(\lambda_n)}{\{K \cdot p^*(T^* + \beta^{-1})\}^{1/(\rho^* - 1)}} \right] \) for arbitrarily large \( n \). Hence we have,
\[
\|a_n\| \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{\{K \cdot p^*(T^* + \alpha^{-1})\}^{1/(\rho^* - 1)}} \right] < \\
\exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{(K \cdot p^*)^{1/(\rho^* - 1)}} \left\{ \frac{1}{(T^* + \alpha^{-1})^{1/(\rho^* - 1)}} - \frac{1}{(T^* + \beta^{-1})^{1/(\rho^* - 1)}} \right\} \right] + \varepsilon
\]
Since $\varepsilon > 0$ is arbitrary and the first term on the right hand side $\to 0$ as $n \to \infty$, we find that the sequence $\{a_n\}$ satisfies (11) and therefore $f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda n}$ belongs to $Y_d$. Using (13) again, we have for $\alpha = 1, 2, \ldots$ 

$$||f_p - f||_\alpha < \varepsilon.$$ 

Hence 

$$d(f_p, f) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{||f_p - f||_\alpha}{1 + ||f_p - f||_\alpha} \leq \frac{\varepsilon}{1 + \varepsilon} \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} < \varepsilon.$$ 

Since the above inequality holds for all $p > n_0$, we finally get $f_p \to f$ as $p \to \infty$ with respect to the metric $d$, where $f \in Y_d$. Hence $Y_d$ is complete. This proves Theorem 1. □

Next we prove

**Theorem 2** A continuous linear transformation $\psi : Y_d \to E$ is of the form 

$$\psi(f) = \sum_{n=1}^{\infty} a_n C_n$$ 

if and only if 

$$|C_n| \leq A \cdot \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{(K \cdot \rho^*(T^* + \alpha^{-1})))^{1/(\rho^*-1)}} \right] \quad \text{for all } n \geq 1, \alpha \geq 1, \quad (14)$$ 

where $A$ is a finite, positive number, $f = f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda n}$ and $\lambda_1$ is sufficiently large.

**Proof.** Let $\psi : Y_d \to E$ be a continuous linear transformation then for any sequence $\{f_m\} \subseteq Y_d$ such that $f_m \to f$, we have $\psi(f_m) \to \psi(f)$ as $m \to \infty$. Now, let $f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda n}$ where $a'_n s \in E$ satisfy (11). Then $f \in Y_d$. Also, let $f_k(s) = \sum_{n=1}^{k} a_n e^{s \lambda n}$. Then $f_k \in Y_d$ for $k = 1, 2, \ldots$. Let $\alpha$ be any fixed positive integer and let $0 < \varepsilon < \alpha^{-1}$. From (11) we can find an integer $m$ such that 

$$||a_n|| < \exp \left[ \frac{-\lambda_n \Phi(\lambda_n)}{(K \cdot \rho^*(T^* + \varepsilon)))^{1/(\rho^*-1)}} \right], \quad \forall n > m.$$
Then
\[
\| f - \sum_{n=1}^{m} a_n e^{s \lambda_n} \|_{\alpha} = \| \sum_{n=m+1}^{\infty} a_n e^{s \lambda_n} \|_{\alpha} \\
= \sum_{n=m+1}^{\infty} \| a_n \| \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{[K \cdot \rho^*(T^* + \alpha^{-1})]^{1/(\rho^*-1)}} \right] \\
< \sum_{n=m+1}^{\infty} \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{(K \cdot \rho^*)^{1/(\rho^*-1)}} \left\{ (T^* + \alpha^{-1})^{-1/(\rho^*-1)} - (T^* + \epsilon)^{-1/(\rho^*-1)} \right\} \right] < \epsilon,
\]
for sufficiently large values of \( m \).

Hence
\[
d (f, f_m) = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{\| f - f_m \|_{\alpha}}{1 + \| f - f_m \|_{\alpha}} \leq \frac{\epsilon}{1 + \epsilon} < \epsilon,
\]
i.e. \( f_m \to f \) as \( m \to \infty \) in \( Y_d \). Since \( \psi \) is continuous, we have
\[
\lim_{m \to \infty} \psi (f_m) = \psi (f).
\]

Let us denote by \( C_n = \psi (e^{s \lambda_n}) \). Then
\[
\psi (f_m) = \sum_{n=1}^{m} a_n \psi (e^{s \lambda_n}) = \sum_{n=1}^{m} a_n C_n.
\]

Also \( |C_n| = |\psi (e^{s \lambda_n})| \). Since \( \psi \) is continuous on \( Y_d \) it is continuous on \( Y_{\| \cdot \|_{\alpha}} \) for each \( \alpha = 1, 2, 3, \ldots \). Hence there exists a positive constant \( A \) independent of \( \alpha \) such that
\[
|\psi (e^{s \lambda_n})| = |C_n| \leq A \| p \|_{\alpha}, \quad \alpha \geq 1
\]
where \( p (s) = e^{s \lambda_n} \). Now using the definition of the norm for \( p (s) \), we get
\[
|C_n| \leq A \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{[K \cdot \rho^*(T^* + \alpha^{-1})]^{1/(\rho^*-1)}} \right], \quad n \geq 1, \quad \alpha \geq 1.
\]

Hence we get \( \psi (f) = \sum_{n=1}^{\infty} a_n C_n \), where the sequence \( \{C_n\} \) satisfies (14).

Conversely, suppose that \( \psi (f) = \sum_{n=1}^{\infty} a_n C_n \) and \( C'_n \)'s satisfy (14). Then for \( \alpha \geq 1 \),
\[
\| \psi (f) \| \leq A \sum_{n=1}^{\infty} \| a_n \| \exp \left[ \frac{\lambda_n \Phi(\lambda_n)}{[K \cdot \rho^*(T^* + \alpha^{-1})]^{1/(\rho^*-1)}} \right]
\]
i.e. $|\psi(f)| \leq A||f||_\alpha \alpha \geq 1$.

Now, since $d(f, g) = \sum_{\alpha \geq 1} \frac{1}{2^\alpha} \cdot \frac{||f-g||_\alpha}{1+||f-g||_\alpha}$, therefore $\psi$ is continuous. This completes the proof of Theorem 2.

3 Linear continuous transformations and proper bases

Following Kamthan and Hussain [2] we give some more definitions. A subspace $X_0$ of $X$ is said to be spanned by a sequence $\{\alpha_n\} \subseteq X$ if $X_0$ consists of all linear combinations $\sum_{n=1}^{\infty} c_n \alpha_n$ such that $\sum_{n=1}^{\infty} c_n \alpha_n$ converges in $X$. A sequence $\{\alpha_n\} \subseteq X$ which is linearly independent and spans a subspace $X_0$ of $X$ is said to be a base in $X$. In particular, if $e_n \in X$, $e_n(s) = e^{\lambda_n}, n \geq 1$, then $\{e_n\}$ is a base in $X$. A sequence $\{\alpha_n\} \subseteq X$ will be called a ‘proper base’ if it is a base and it satisfies the condition:

“For all sequences $\{\alpha_n\} \subseteq E$, convergence of $\sum_{n=1}^{\infty} ||\alpha_n|| \alpha_n$ in $X$ implies the convergence of $\sum_{n=1}^{\infty} \alpha_n e_n$ in $X'$. As defined above, for $f \in Y$, we put $||f, T^* + \delta|| = \sum_{n=1}^{\infty} ||\alpha_n|| \exp \left[ \frac{\lambda_n \psi(\lambda_n)}{[K_n \psi(T^* + \delta)]^{1/\rho^* - 1}} \right] \cdot \exp$. We now prove

**Theorem 3** A necessary and sufficient condition that there exists a continuous linear transformation $F : Y \rightarrow Y$ with $F(e_n) = \alpha_n, n = 1, 2, \ldots$, where $\alpha_n \in Y$, is that for each $\delta > 0$

$$\lim_{n \rightarrow \infty} \sup_{\alpha_n} \frac{\log ||\alpha_n : T^* + \delta||^{1/\lambda_n}}{\psi(\lambda_n)} \leq \left( \frac{\rho^* - 1}{\rho^*} \right) (\rho^* T^*)^{-1/\rho^* - 1}. \quad (15)$$

**Proof.** Let $F$ be a continuous linear transformation from $Y$ into $Y$ with $F(e_n) = \alpha_n, n = 1, 2, \ldots$. Then for any given $\delta > 0$, there exists $\delta_1 > 0$ and a constant $K' = K'(\delta)$ depending on $\delta$ only, such that

$$||F(e_n); T^* + \delta|| \leq K'||e_n; T^* + \delta_1|| \Rightarrow ||\alpha_n; T^* + \delta||$$

$$\leq K' \exp \left\{ \frac{(\rho^* - 1)\lambda_n \varphi(\lambda_n)}{[T^* + \delta_1]^{1/\rho^* - 1}} (\rho^* \rho^* / \rho^* - 1) \right\}$$

$$\Rightarrow \log ||\alpha_n; T^* + \delta||^{1/\lambda_n}$$

$$\leq o(1) + \frac{\varphi(\lambda_n) (\rho^* - 1)}{[T^* + \delta_1]^{1/\rho^* - 1} (\rho^* \rho^* / \rho^* - 1)}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sup_{\alpha_n} \frac{\log ||\alpha_n; T^* + \delta||^{1/\lambda_n}}{\varphi(\lambda_n)} \leq \frac{(\rho^* - 1)}{\rho^* (\rho^* T^*)^{1/\rho^* - 1}}.$$
Conversely, let the sequence \( \{ \alpha_n \} \) satisfy (15) and let \( \alpha = \sum_{n=1}^{\infty} a_n \varepsilon_n \). Then we have
\[
\lim_{n \to \infty} \sup_{\lambda_n} \frac{\lambda_n}{\log \| \alpha_n \|^{-1/\lambda_n}} \leq \frac{\rho^* (\rho^* T)^{1/\rho^*-1}}{(\rho^*-1)}.
\]
Hence, given \( \eta > 0 \), there exists \( N_0 = N_0 (\eta) \), such that
\[
\frac{\varphi (\lambda_n)}{\log \| \alpha_n \|^{-1/\lambda_n}} \leq \frac{\rho^*}{(\rho^*-1)} \{ \rho^* (T^* + \eta) \}^{1/\rho^*-1} \quad \forall \ n \geq N_0.
\]
Further, for a given \( \eta_1 > \eta \), from (15), we can find \( N_1 = N_1 (\eta_1) \) such that for \( n \geq N_1 \)
\[
\log \| \alpha_n ; T^* + \delta \|^{1/\lambda_n} \frac{1}{\varphi (\lambda_n)} \leq \left( \frac{T^* + \eta}{T^* + \eta_1} \right)^{1/(\rho^*-1)}.
\]
Choose \( n \geq \max (N_0, N_1) \). Then
\[
\Rightarrow \| \alpha_n \| \| \alpha_n ; T^* + \delta \| \leq \| \alpha_n \|_{1-(T^*+\eta/T^*+\eta_1)^{1/(\rho^*-1)}} = \| \alpha_n \|^\beta \quad \text{(say)}
\]
where \( \beta = 1 - (T^* + \eta/T^* + \eta_1)^{1/(\rho^*-1)} > 0 \). Now from (5) we can easily show that for any arbitrary large number \( K > 0, \| \alpha_n \| < e^{-K \lambda_n} \).

Hence we have for all large values of \( n, \| \alpha_n \| \| \alpha_n ; T^* + \delta \| \leq e^{-K \lambda_n} \).

Consequently the series \( \sum_{n=1}^{\infty} \| \alpha_n \| \| \alpha_n ; T^* + \delta \| \) converges for each \( \delta > 0 \). Therefore \( \sum_{n=1}^{\infty} \| \alpha_n \| \| \alpha_n \| \) converges to an element of \( Y \). For each \( \alpha \in Y \), We define \( F(\alpha) = \sum_{n=1}^{\infty} a_n \alpha_n \). Then \( F(\varepsilon_n) = \alpha_n \). Now, given \( \delta > 0, \exists \delta_1 > 0 \) such that
\[
\frac{\log \| \alpha_n \| \| \alpha_n ; T^* + \delta \|^{1/\lambda_n}}{\varphi (\lambda_n)} \leq \left( \frac{\rho^* - 1}{\rho^*} \right) \{ \rho^* (T^* + \eta_1) \}^{-1/(\rho^*-1)}
\]
for all \( n \geq N = N(\delta, \delta_1) \). Hence
\[
\Rightarrow \| \alpha_n ; T^* + \delta \| \leq K' \exp \left\{ \frac{(\rho^* - 1) \lambda_n \varphi (\lambda_n)}{\rho^* (\rho^* (T^* + \delta_1))^{1/\rho^*-1}} \right\}
\]
where \( K' = K' (\delta) \) and the inequality is true for all \( n > 0 \). Now
\[
\| F(\alpha) ; T^* + \delta \| \leq \sum_{n=1}^{\infty} \| \alpha_n \| \| \alpha_n ; T^* + \delta \|
\]
\[
\leq K' \sum_{n=1}^{\infty} \| \alpha_n \| \exp \left\{ \frac{(\rho^* - 1) \lambda_n \varphi (\lambda_n)}{\rho^* (\rho^* (T^* + \delta_1))^{1/\rho^*-1}} \right\} = K' \| \alpha_n ; T^* + \delta \|.
\]
Hence $F$ is continuous. This proves Theorem 3. \hfill \Box

We now give some results characterizing the proper bases.

**Lemma 1** In the space $Y_d$, the following three conditions are equivalent:

(i) For each $\delta > 0$, \( \lim_{n \to \infty} \sup \frac{\log \|a_n, T^* + \delta\|^1/\lambda_n}{\varphi(\lambda_n)} \leq \left( \frac{p^*-1}{p^*} \right) \left( \rho^*(T^*) - 1/(p^*-1) \right) \).

(ii) For any sequence \( \{a_n\} \) in $E$, the convergence of \( \sum_{n=1}^{\infty} a_n e_n \) in $Y$ implies that \( \lim_{n \to \infty} \|a_n\| \alpha_n = 0 \) in $Y$.

(iii) For any sequence \( \{a_n\} \) in $E$, the convergence of \( \sum_{n=1}^{\infty} a_n e_n \) in $Y$ implies the convergence of \( \sum_{n=1}^{\infty} \|a_n\| \alpha_n \) in $Y$.

**Proof.** First suppose that (ii) holds. Then for any sequence \( \{a_n\} \sum_{n=1}^{\infty} a_n e_n \) converges in $Y$ implies that \( \sum_{n=1}^{\infty} \|a_n\| \alpha_n \) converges in $Y$ which in turn implies that \( \|a_n\| \alpha_n \to 0 \) as $n \to \infty$. Hence (ii) $\Rightarrow$ (iii).

Now we assume that (iii) is true but (i) is false. Hence for some $\delta > 0$, there exists a sequence \( \{n_k\} \) of positive integers such that $\forall n_k$, $k = 1, 2, \ldots$,

\[
\frac{\log \|a_{n_k}, T^* + \delta\|^1/\lambda_{n_k}}{\varphi(\lambda_{n_k})} > \left( \frac{p^*-1}{p^*} \right) \left( \rho^*(T^* + \frac{1}{k}) \right)^{-1/(p^*-1)}.
\]

Define a sequence \( \{a_n\} \) as follows:

\[
\|a_n\| = \begin{cases} \|a_{n_k}, T^* + \delta\|^{-1}, & n = n_k \\ 0, & n \neq n_k \end{cases}
\]

Then, we have for all large values of $k$,

\[
\frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}\|^{-1/\lambda_{n_k}}} = \frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}, T^* + \delta\|^1/\lambda_{n_k}} < \left( \frac{\rho^*}{\rho^* - 1} \right) \left( \rho^*(T^* + \frac{1}{k}) \right)^{1/(p^*-1)}.
\]

Hence,

\[
\lim_{k \to \infty} \sup_n \frac{\varphi(\lambda_{n_k})}{\log \|a_{n_k}\|^{-1/\lambda_{n_k}}} \leq \left( \frac{\rho^*}{\rho^* - 1} \right) (\rho^*(T^*))^{1/(p^*-1)}.
\]

Thus \( \{a_n\} \) defined by (16) satisfies the condition

\[
\lim_{n \to \infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} \leq \left( \frac{\rho^*}{\rho^* - 1} \right) (\rho^*(T^*))^{1/(p^*-1)}
\]

which in view of Theorem 1 above is equivalent to the condition that \( \sum a_n e_n \) converges in $Y$. Hence by (iii), \( \lim_{n \to \infty} \|a_n\| \alpha_n = 0 \). However

\[
\|a_{n_k} \| \alpha_{n_k}, T^* + \delta \| = \|a_{n_k}\| \cdot \|\alpha_{n_k}, T^* + \delta\| = 1.
\]
Hence \( \lim_{n \to \infty} \|a_n\| \neq 0 \) in \( Y(\rho^*, T^*, \delta) \). This is a contradiction. Hence (iii) \( \Rightarrow \) (i).

In the course of proof of Theorem 3 above, we have already proved that (i) \( \Rightarrow \) (ii). Thus the proof of Lemma 1 is complete. \( \Box \)

Next we prove

**Lemma 2** The following three properties are equivalent:

(a) For all sequences \( \{a_n\} \) in \( E \), \( \lim_{n \to \infty} a_n\alpha_n = 0 \) in \( Y \) implies that \( \sum_{n=1}^{\infty} a_n e_n \) converges in \( Y \).

(b) For all sequences \( \{a_n\} \) in \( E \), the convergence of \( \sum_{n=1}^{\infty} \|a_n\| \alpha_n \) in \( Y \) implies the convergence of \( \sum_{n=1}^{\infty} a_n e_n \).

(c) \( \lim_{\delta \to 0} \left\{ \lim_{n \to \infty} \left( \frac{\log \|a_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \right) \right\} \geq \left( \frac{\rho^* - 1}{\rho^*} \right) (\rho^* T^*)^{-1/(\rho^* - 1)} \).

**Proof.** Obviously (a) \( \Rightarrow \) (b). We now prove that (b) \( \Rightarrow \) (c). To prove this, we suppose that (b) holds but (c) does not hold. Hence

\[
\lim_{\delta \to 0} \left\{ \lim_{n \to \infty} \left( \frac{\log \|a_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \right) \right\} < \left( \frac{\rho^* - 1}{\rho^*} \right) (\rho^* T^*)^{-1/(\rho^* - 1)}. 
\]

Since \( \log \|a_n; T + \delta\| \) increases as \( \delta \) decreases, this implies that for each \( \delta > 0 \),

\[
\left\{ \lim_{n \to \infty} \frac{\log \|a_n; T^* + \delta\|^{1/\lambda_n}}{\varphi(\lambda_n)} \right\} < \left( \frac{\rho^* - 1}{\rho^*} \right) (\rho^* T^*)^{-1/(\rho^* - 1)}. 
\]

Hence, if \( \eta > 0 \) be a fixed small positive number, then for each \( r > 0 \), we can find a positive number \( n_r \) such that \( \forall r \), we have \( n_{r+1} > n_r \) and

\[
\lim_{n \to \infty} \frac{\log \|a_{n_r}; T^* + r^{-1}\|^{1/\lambda_{n_r}}}{\varphi(\lambda_{n_r})} < \left( \frac{\rho^* - 1}{\rho^*} \right) (\rho^* (T^* + \eta))^{-1/(\rho^* - 1)}. 
\] (17)

Now we choose a positive number \( \eta_1 < \eta \), and define a sequence \( \{a_n\} \) as

\[
\|a_n\| = \begin{cases} \left( \frac{T^* + \eta_1}{T^* + \eta} \right)^{\lambda_n} \exp \left\{- \left( \frac{\rho^* - 1}{\rho^*} \right) \frac{\lambda_n \varphi(\lambda_n)}{\rho^* (T^* + \eta_1)^{1/(\rho^* - 1)}} \right\}, & n = n_r, \\ 0, & n \neq n_r. \end{cases}
\]

Then, for any \( \delta > 0 \)

\[
\sum_{n=1}^{\infty} \|a_n\| \cdot \|\alpha_n; T^* + \delta\| = \sum_{r=1}^{\infty} \|a_{n_r}\| \cdot \|\alpha_{n_r}; T^* + \delta\|. 
\] (18)
For any given $\delta > 0$, we omit from the above series those finite number of terms, which correspond to those number $n_k$ for which $1/r$ is greater than $\delta$. The remainder of the series in (18) is dominated by $\sum_{r=1}^{\infty} \|a_n\| \cdot \|a_n\| T^{r+1}$. Now by (17) and (18), we find that

$$\sum_{r=1}^{\infty} \|a_n\| \cdot \|a_n\| T^{r+1} \leq \sum_{r=1}^{\infty} \left\{ \exp \left\{ - \left( \frac{\rho^* - 1}{\rho^*} \right) \frac{\lambda_n \varphi(\lambda_n)}{\varphi(\lambda_n)} \right\} \frac{\lambda_n \varphi(\lambda_n)}{\varphi(\lambda_n)} \right\},$$

which contradicts (10). This proves (b) $\Rightarrow$ (c).

Now we prove that (c) $\Rightarrow$ (a). We assume (c) is true but (a) is not true. Then there exists a sequence $\{a_n\}$ of complex numbers for which $\|a_n\| a_n \rightarrow 0$ in $Y$, but $\sum_{n=1}^{\infty} a_n e_n$ does not converge in $Y$. This implies that

$$\lim_{n \rightarrow \infty} \sup \frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} > \left( \frac{\rho^*}{\rho^* - 1} \right) \left( \rho^* T^* \right)^{1/(\rho^*-1)},$$

Hence there exists a positive number $\varepsilon$ and a sequence $\{n_k\}$ of positive integers such that

$$\frac{\varphi(\lambda_n)}{\log \|a_n\|^{-1/\lambda_n}} = \left( \frac{\rho^*}{\rho^* - 1} \right) \left( \rho^* T^* + \varepsilon \right)^{1/(\rho^*-1)}, \quad \forall n = n_k.$$

We choose another positive number $\eta < \varepsilon/2$. By assumption we can find a positive number $\delta$ i.e. $\delta = \delta(\eta)$ such that

$$\lim_{n \rightarrow \infty} \inf \frac{\log \|a_n\| T^{r+1/\lambda_n}}{\varphi(\lambda_n)} \geq \left( \frac{\rho^* - 1}{\rho^*} \right) \left( \rho^* T^* + \eta \right)^{-1/(\rho^*-1)}.$$
Hence there exists $N = N(\eta)$, such that
\[
\frac{\log ||\alpha_n, T^* + \delta||}{\varphi(\lambda_n)} \geq \left( \frac{\rho^* - 1}{\rho^*} \right) (\rho^*(T^* + 2\eta))^{-1/(\rho^* - 1)}, \quad \forall n \geq N.
\]
Therefore
\[
\max \{ ||a_n|| \alpha_n, T^* + \delta \} = \max \{ ||a_n|| \cdot ||\alpha_n, T^* + \delta|| \} \\
\geq \max \{ ||a_n|| \cdot ||\alpha_n; T^* + \delta|| \} \\
\geq \exp \left\{ -\frac{\lambda_n \varphi(\lambda_n)(\rho^* - 1)}{\rho^* \{(\rho^*(T^* + \epsilon))^1/(\rho^* - 1)\}} \right\} \\
\times \exp \left\{ \frac{\lambda_n \varphi(\lambda_n)(\rho^* - 1)}{\rho^* \{(\rho^*(T^* + 2\eta))^1/(\rho^* - 1)\}} \right\} > 1
\]
for $n_k > N$ as $\epsilon > 2\eta$.

Thus $\{ ||a_n|| \alpha_n \}$ does not tend to zero in $Y(\rho^*, T^*, \delta)$ for the $\delta$ chosen above. Hence $\{ ||a_n|| \alpha_n \}$ does not tend to 0 in $Y$ and this is a contradiction. Thus (c)$\Rightarrow$(a) is proved. This proves Lemma 2.

Lastly we prove:

**Theorem 4** A base $\{ \alpha_n \}$ in a closed subspace $Y_0$ of $Y$ is proper if and only if the conditions (i) and (c) stated above are satisfied.

**Proof.** Let $\{ \alpha_n \}$ be a proper base in a closed subspace $Y_0$ of $Y$. Hence for any sequence of complex number $\{ a_n \}$ the convergence of $\sum_{n=1}^{\infty} ||a_n|| \alpha_n$ in $Y_0$ implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in $Y_0$. Therefore (b) and hence (c) is satisfied. Further the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in $Y_0$ is equivalent to the condition
\[
\lim_{n \to \infty} \sup \frac{\varphi(\lambda_n)}{\log ||a_n||^{-1/\lambda_n}} = \left( \frac{\rho^*}{\rho^* - 1} \right) (\rho^* T^*)^{1/(\rho^* - 1)}.
\]
Now let $\alpha = \sum_{n=1}^{\infty} a_n e_n$. Then proceeding as in second part of the proof of Theorem 1, we can prove that $\sum_{n=1}^{\infty} ||a_n|| \alpha_n$ converges to an element of $Y_0$ and thus (ii) is satisfied. But (ii) is equivalent to (i). Hence necessary part of the theorem is proved.

Conversely, suppose that conditions (i) and (c) are satisfied, with $\{ \alpha_n \}$ being a base in a closed subspace $Y_0$ of $Y$. Then by Lemma 2, we find that for any sequence $\{ a_n \}$ in $E$, convergence of $\sum_{n=1}^{\infty} ||a_n|| \alpha_n$ in $Y_0$ implies the convergence of $\sum_{n=1}^{\infty} a_n e_n$ in $Y_0$. Therefore $\{ \alpha_n \}$ is a proper base of $Y_0$. This concludes the proof. \( \square \)
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References


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