A Note on Spectra of Weighted Composition Operators on Weighted Banach Spaces of Holomorphic Functions

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Abstract. We characterize the spectra of bounded weighted composition operators acting on the weighted Banach spaces $H_v^\infty$ of analytic functions on the unit disc defined for a radial weight $v$, when the symbol of the operator has a fixed point in the open unit disc.

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1. Introduction

Let $v$ be a strictly positive continuous function (weight) on the open unit disc $D$. We consider

$$H_v^\infty := \{ f \in H(D); \|f\|_v := \sup_{z \in D} v(z)|f(z)| < \infty \}$$

and

$$H_v^0 := \{ f \in H_v^\infty; \lim_{|z| \to 1} v(z)|f(z)| = 0 \}$$

defined with norm $\|\cdot\|_v$ where $H(D)$ denotes the space of holomorphic functions on $D$. For more information on this type of spaces we refer the reader to [2], [4] and [11].

Let $\varphi : D \to D$ be an analytic self map of the unit disc. Each such map induces through composition a linear composition operator $C_\varphi(f) = f \circ \varphi$ acting on this type of spaces. If we choose $\psi \in H(D)$ we obtain a weighted composition operator $C_{\varphi,\psi}(f) = \psi(f \circ \varphi)$.

Composition operators have been studied on various spaces of analytic functions. We refer the reader to the monographs [7] and [14]. The case of operators defined on weighted Banach spaces of the type defined above was treated, for example in [4], [3] and [6]. The spectrum of a composition operator has recently been investigated in [8], [15] and [12] as well as in [1] and [5]. More precisely, Aron and Lindström

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determined the spectrum of a bounded weighted composition operator acting on weighted Banach spaces $H_v^\infty$ where $v_p(z) = (1 - |z|^2)^p$, $p > 0$ (see [1]). This was generalized by Bonet, Galindo and Lindström in the sense that they were able to characterize the spectrum of a bounded composition operator defined on a weighted Banach space $H_v^\infty$, where $v$ is a typical weight. The aim of this article is to generalize the results of [1] and [5], i.e. to give the spectrum of a bounded weighted composition operator acting on weighted Banach spaces $H_v^\infty$ of holomorphic functions using methods of [1] and [5].

2. Notations and Definitions

We refer the reader to [7] and [14] for notation on composition operators. The closed unit ball of $H_v^\infty$ (resp. $H_v^0$) is denoted by $B_v^\infty$ (resp. $B_v^0$). Many results on weighted spaces of analytic functions and on operators between them have to be given in terms of the so-called associated weights (see [2]) and in terms of the weights. For a weight $v$ the associated weight $\tilde{v}$ is defined as follows

$$\tilde{v}(z) := \frac{1}{\sup \{ |f(z)|; f \in H_v^\infty, \|f\|_v \leq 1 \}} = \frac{1}{\|\delta_z\|_{H_v^\infty}}, \quad z \in D,$$

where $\delta_z$ denotes the point evaluation of $z$. The associated weights are also continuous and $\tilde{v} \geq v > 0$ (see [2]). Furthermore, for each $z \in D$ there is $f_z \in H_v^\infty$, $\|f_z\|_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. A weight $v$ is called essential if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for every } z \in D.$$

For examples of essential weights and conditions when weights are essential, see [2], [4] and [3]. Especially interesting are radial weights $v$, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. Every radial weight $v$ which is non-increasing with respect to $|z|$ and such that $\lim_{|z| \to 1} v(z) = 0$ is called a typical weight. In the sequel every radial weight is assumed to be non-increasing. For typical weights $v$ the polynomials lie dense in $H_v^0$.

The essential spectrum $\sigma_{e,X}(T)$ of a bounded operator $T$ on the Banach space $X$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Fredholm. It is known that $\sigma_{e,X}(T) = \sigma_{e,X}(T^*)$, see [9]. The essential spectral radius is given by

$$r_{e,X}(T) = \sup \{ |\lambda|; \lambda \in \sigma_{e,X}(T) \}.$$

Another way of expressing the essential spectral radius is

$$r_{e,X}(T) = \lim_n \|T^n\|^\frac{1}{n}_{e,X}$$

where $\|T\|_{e,X}$ denotes the essential norm of $T$, i.e. the distance from the compact operators on $X$,

$$\|T\|_{e,X} = \inf \{ \|T - K\|; K \text{ is a compact operator on } X \}.$$

Let $\lambda \in \sigma_X(T)$ be such that $|\lambda| > r_{e,X}(T)$. Then $\lambda$ lies in the unbounded component of $\mathbb{C} \setminus \sigma_{e,X}(T)$. Now the Fredholm theory gives that $\lambda$ is an isolated point of $\sigma_X(T)$ which also is an eigenvalue of finite multiplicity. We recall the following well-known result (see [9]):
Lemma 2.1. Let $T : X \rightarrow X$ be a bounded operator. If $\lambda \in \sigma_X(T)$ is such that $|\lambda| > r_{e,X}(T)$, then $\lambda$ is an isolated eigenvalue of finite multiplicity.

By [6] Proposition 3.1 weighted composition operators $C_{\varphi,\psi} : H^\infty_v \rightarrow H^\infty_v$ are continuous if and only if
\[
\sup_{z \in D} |\psi(z)| \frac{v(z)}{\overline{v}(\varphi(z))} < \infty.
\]
For a similar result see also [13]. If $\|C_{\varphi,\psi}\|_{e,v}$ denotes the essential norm of $C_{\varphi,\psi}$, then we get
\[
\|C_{\varphi,\psi}\|_{e,v} = \lim_{r \rightarrow \infty} \sup_{|\varphi(z)| > r} |\psi(z)| \frac{v(z)}{\overline{v}(\varphi(z))}.
\]
(See [6] and [13].)

3. Results

In the case of composition operators the following lemma was given by Bonet, Galindo and Lindström in [5].

Lemma 3.1. Let $v$ be a typical weight and $\varphi(0) = 0$ as well as $0 < |\varphi'(0)| < 1$. Moreover, we assume that $C_{\varphi,\psi}$ is bounded. Then $\sigma_{H^\infty_v}(C_{\varphi,\psi})$ contains $\psi(0)\varphi'(0)^n$ for non-negative integers $n$.

Proof. If $\psi(0) = 0$, then $\psi(0)\varphi'(0)^n \in \sigma_{H^\infty_v}$. Thus we can assume that $\psi(0) \neq 0$. We use an argument of [10] to show that $z^n \in H^\infty_v$, $n \in \mathbb{N}$, is not in the range of $C_{\varphi,\psi} - \psi(0)\varphi'(0)^n I$ on $H^\infty_v$. Let us assume that $f \in H^\infty_v$ and
\[
f(\varphi(z))\psi(z) - \varphi'(0)\psi(0)f(z) = z.
\]
Then, we obtain $f(0)\psi(0) - \varphi'(0)\psi(0)f(0) = 0$. Since $\psi(0) \neq 0$ and $0 < |\varphi'(0)| < 1$ we obtain $f(0) = 0$. Now differentiation on both sides yields
\[
\psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z)) - \varphi'(0)\psi(0)f'(z) = 1.
\]
With $z = 0$ we obtain $0 = \psi(0)f'(0)\varphi'(0) - \varphi'(0)\psi(0)f'(0) = 1$ which is a contradiction. For $n > 1$, suppose $f \in H^\infty_v$ and $\psi(z)f(\varphi(z)) - \psi(0)\varphi'(0)^n f(z) = z^n$. By repeated differentiation on both sides we get for all $k < n$ that $f^{(k)}(0) = 0$. Then for $k = n$ and $z = 0$ we obtain the contradiction $0 = n!$. This means that $\psi(0)\varphi'(0)^n \in \sigma_{H^\infty_v}(C_{\varphi,\psi})$ for all $n > 0$. If $n = 0$, then $\psi(0) \in \sigma_{H^\infty_v}(C_{\varphi,\psi})$, since $C_{\varphi,\psi} - \psi(0)I$ is not onto.

For a positive integer $m$ and an arbitrary weight $v$, let us consider the closed subspace $H^\infty_{v,m}$ of $H^\infty_v$ defined by
\[
H^\infty_{v,m} := \{ f \in H^\infty_v; f \text{ has a zero of at least order } m \text{ at } 0 \}.
\]
Let $\| . \|_{m,v}$ be the induced norm on $H^\infty_{v,m}$. The following result can be found in [5]. For standard weights, the result was obtained by Aron and Lindström in [1].

Proposition 3.1. Let $m \in \mathbb{N}$ and $v$ be an arbitrary weight. Then there exists a constant $M_m > 0$ such that
\[
|f(w)| \leq M_m \frac{1}{v(w)} \|f\|_v |w|^m,
\]
for all $f \in H^\infty_{v,m}$ and $w \in D$. 
Lemma 3.2. Let \( v \) be a typical weight. Suppose that \( \varphi(0) = 0 \) and that \( C_{\varphi,\psi} \) is bounded on \( H_v^\infty \). If \( \lambda \neq 0 \) is an eigenvalue of \( C_{\varphi,\psi} \), then \( \lambda \in \{ \psi(0)\varphi'(0)^n \}_{n=0}^\infty \).

Proof. Suppose that \( \lambda \neq 0 \) is an eigenvalue of \( C_{\varphi,\psi} \) with the corresponding eigenvector \( f \neq 0 \). Then for all \( z \in \mathbb{D} \), we have

\[
C_{\varphi,\psi}f(z) = \psi(z)f(\varphi(z)) = \lambda f(z).
\]

We obviously get \( \psi(0)f(0) = \lambda f(0) \). If \( f(0) \neq 0 \), \( \lambda = \psi(0) \) follows. If \( f(0) = 0 \) by differentiation on both sides of the equation we obtain:

\[
\psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z)) = \lambda f'(z).
\]

Since \( f(0) = 0 \), we obtain \( \psi(0)\varphi'(0)f(0) = \lambda f'(0) \). If \( f'(0) \neq 0 \), then \( \lambda = \psi(0) \varphi'(0) \).

If \( f'(0) = 0 \) we proceed by induction until we have found a \( k \) such that \( f^{(k)}(0) \neq 0 \) which must exist. In that case we obtain \( \lambda = \psi(0)\varphi'(0)^k \). Thus, the claim follows.

We need the following lemma which is taken from [7].

Lemma 3.3. If \( \varphi \) is not an automorphism and \( \varphi(0) = 0 \), then given \( 0 < r < 1 \), there exists \( 1 \leq M < \infty \) such that if \( \{z_k\}_{k=-K}^{\infty} \) is an iteration sequence with \( \|z_n\| \geq r \) for some non-negative integer \( n \) and \( \{w_k\}_{k=-K}^{\infty} \) are arbitrary numbers, then there exists \( f \in H^\infty \) with \( f(z_k) = w_k, -K \leq k \leq n \) and \( \|f\| \leq M \sup\{ |w_k| : -K \leq k \leq n \} \).

Further there exists \( b < 1 \) such that for any iteration sequence \( (z_k) \), i.e. for any sequence \( (z_k) \) with \( z_{k+1} = \varphi(z_k) \) we have \( |z_{k+1}|/|z_k| \leq b \) whenever \( |z_k| \leq 1/2 \).

The proof of the following theorem was inspired by [1] and [5].

Theorem 3.1. Let \( v \) be a typical weight. Suppose \( \varphi \), not an automorphism has fixed point \( a \in D \) and that \( C_{\varphi,\psi} : H_v^\infty \to H_v^\infty \) is bounded. Then

\[
\sigma_{H_v^\infty}(C_{\varphi,\psi}) = \{ \lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^\infty}(C_{\varphi,\psi}) \} \cup \{ \psi(a)\varphi'(a)^n \}_{n=0}^\infty.
\]

Proof. We can assume that \( a = 0 \). If \( a \neq 0 \), we consider

\[
\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad \psi_1(z) = \psi \circ \varphi_a, \quad \text{and} \quad \varphi_1 = \varphi_a \circ \varphi \circ \varphi_a.
\]

Then \( \varphi_1(0) = 0, \psi_1(0) = \psi(a), \varphi'_1(0) = \varphi'(a) \) and

\[
C_{\varphi_a} \circ C_{\varphi_1,\psi_1} \circ C_{\varphi_a}^{-1} = C_{\varphi,\psi}.
\]

Hence, \( C_{\varphi,\psi} \) and \( C_{\varphi_1,\psi_1} \) are similar. Therefore, they have the same spectrum and the same essential spectral radius.

By Lemma 3.1, we have that \( \{ \psi(0)\varphi'(0)^n \}_{n=0}^\infty \subset \sigma_{H_v^\infty}(C_{\varphi,\psi}). \) If \( \lambda \in \sigma_{H_v^\infty}(C_{\varphi,\psi}) \) and \( |\lambda| > r_{e,H_v^\infty}, \) then Lemma 2.1 yields, that \( \lambda \) is an eigenvalue. If \( \lambda \neq 0 \) is an eigenvalue of \( C^{\varphi,\psi} \), then by Lemma 3.2, we can find \( n \) such that \( \lambda = \psi(0)\varphi'(0)^n \).

It remains to show that

\[
\{ \lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^\infty}(C_{\varphi,\psi}) \} \subset \sigma_{H_v^\infty}(C_{\varphi,\psi}).
\]

If \( r_{e,H_v^\infty}(C_{\varphi,\psi}) = 0 \), then we have \( 0 \in \sigma_{e,H_v^\infty}(C_{\varphi,\psi}) \subset \sigma_{H_v^\infty}(C_{\varphi,\psi}). \) So, we assume that \( \rho := r_{e,H_v^\infty}(C_{\varphi,\psi}) > 0. \) Since \( \varphi(0) = 0 \) we have that \( \varphi(z) = z\phi(z) \) with \( \phi \in H^\infty \).

Hence \( H^{\infty}_{\varphi,\psi} \) is an invariant subspace under \( C_{\varphi,\psi}. \) Further, \( H^{\infty}_{\varphi,\psi} \) has finite codimension in \( H^{\infty}_{\psi}. \) Now Lemma 7.17 in [7] which is also valid for Banach spaces gives that \( \sigma_{H^{\infty}_{\varphi,\psi}}(C_{\varphi,\psi}) \subset \sigma_{H^{\infty}_{\psi}}(C_{\varphi,\psi}). \) So it is enough to show that any \( \lambda \) with \( 0 < |\lambda| < \rho \) belongs to \( \sigma_{H^{\infty}_{\varphi,\psi}}(C_{\varphi,\psi}) \) for some \( m \) to be found. Let \( C_m \) denote
the restriction of $C_{\varphi,\psi}$ to the invariant closed subspace $H_{v,m}^\infty$. Since $C_m - \lambda I$ is not invertible if $(C_m - \lambda I)^*$ is not bounded from below, we just need to find $m$ with $(C_m - \lambda I)^*$ not bounded from below. Let $1 \leq M < \infty$ be the constant in Lemma 3.3 for $r = 1/4$. We will denote iteration sequences by $\xi = (z_k)_{k=\infty}^{\infty}$ with $K > 0$ and $|z_0| \geq 1/2$. Let $n := \max\{k; |z_k| \geq 1/4\}$. Then $n \geq 0$ and $|z_k| < 1/4$ for $k > n$. By Lemma 3.3 there is $b < 1$ with $|z_{k+1}/z_k| \leq b$ for all $k \geq n$. We may assume that $1/2 < b < 1$. This implies

$$|z_k| \leq b^{k-n}|z_n| \text{ for } k \geq n.$$  

Since $\psi \in H(D)$ is continuous, $0 \leq C := \max\{\sup_{|z|\leq 1/4} |\psi(z)|, |\psi(z_0)|\} < \infty$. We now choose $m$ so large that

$$\frac{b^m C}{|\lambda|} < \frac{1}{2}. \quad (3.2)$$

For every such iteration sequence $\xi = (z_k)_{k=\infty}^{\infty}$ we define the linear functional $L_{\xi,\psi}$ of $H_{v,m}^\infty$ by

$$L_{\xi,\psi}(f) = \sum_{k=\infty}^{\infty} \lambda^{-k} \psi(z_{-K}) \cdots \psi(z_{k-1})f(z_k),$$

where we agree that $\psi(z_{-K}) \cdots \psi(z_{K-1}) = 1$ in the first term of the sum. Indeed $L_{\xi,\psi}$ is bounded. Proposition 3.1 yields

$$\left| \sum_{k=\infty}^{\infty} \lambda^{-k} \psi(z_{-K}) \cdots \psi(z_{k-1})f(z_k) \right|$$

$$\leq M_m |f|_v \left\{ \sum_{k=\infty}^{n} |\lambda|^{-k} |\psi(z_{-K})| \cdots |\psi(z_{k-1})||z_k|^m \tilde{v}(z_k)^{-1} \right.$$  

$$+ |\psi(z_{-K})| \cdots |\psi(z_{n-1})| \sum_{k=n+1}^{\infty} |\lambda|^{-k} |\psi(z_{n})| \cdots |\psi(z_{k-1})||z_k|^m \tilde{v}(z_k)^{-1} \right\}. \quad (3.1)$$

Since $|z_k| < \frac{1}{4}$ for $k > n$ and $\tilde{v}$ is continuous, there is a constant $c > 0$ such that $\tilde{v}(z_k)^{-1} \leq c$ for $k > n$. Further applying (3.1) and (3.2) we get

$$\sum_{k=n+1}^{\infty} |\lambda|^{-k} |\psi(z_{n})| \cdots |\psi(z_{k-1})||z_k|^m \tilde{v}(z_k)^{-1} \leq \sum_{k=n+1}^{\infty} |\lambda|^{-k} C^{k-n} |z_k|^m c$$

$$\leq c |\sum_{k=n+1}^{\infty} \frac{|z_n|}{|\lambda|^n} \sum_{k=n+1}^{\infty} \left( \frac{b^m C}{|\lambda|} \right)^{k-n} < \infty.$$  

Thus $\psi(z_{-K}) \cdots |\psi(z_{n-1})| \sum_{k=n+1}^{\infty} |\lambda|^{-k} |\psi(z_{n})| \cdots |\psi(z_{k-1})||z_k|^m \tilde{v}(z_k)^{-1} < \infty$, and $L_{\xi,\psi}$ is bounded. Let us find next a lower bound for $\|L_{\xi,\psi}\|_{v,m}$. There exists $f_{z_0} \in H^\infty$ with $\|f_{z_0}\|_v \leq 1$, so that $|f_{z_0}(z_0)| = 1/\tilde{v}(z_0)$. By Lemma 3.3, there is $f_1 \in H^\infty$ with $\|f_1\|^{\infty} \leq M$, satisfying $|f_1(z_0)| = 1$, $z_0^m f_1(z_0)f_{z_0}(z_0) > 0$ and $f_1(z_k) = 0$ for $-K \leq k \leq n$, $k \neq 0$. Now, the function $g(z) := z^m f_1(z)f_{z_0}(z)$
belongs to $H_{v,m}^\infty$ and $\|g\|_v \leq M$. Further
\[
L_{\xi,\psi}(g) = \psi(z_{-K}) \cdots \psi(z_{-1}) z_0^m f_1(z_0) f_0(z_0) \\
+ \sum_{k=n+1}^{\infty} \lambda^{-k} \psi(z_{-K}) \cdots \psi(z_{k-1}) z_k^m f_1(z_k) f_0(z_k).
\]
(3.3)

Since $\tilde{v}$ is not increasing, we get using again (3.1) and (3.2)
\[
\left| \sum_{k=n+1}^{\infty} \lambda^{-k} \psi(z_{-K}) \cdots \psi(z_{k-1}) z_k^m f_1(z_k) f_0(z_k) \right| \\
\leq M |\psi(z_{-K})| \cdots |\psi(z_{n-1})| \frac{z_n^m}{|\lambda|^n} \sum_{k=n+1}^{\infty} \left( \frac{b^m C}{|\lambda|} \right)^{k-n} \frac{1}{\tilde{v}(z_n)} \\
\leq M |\psi(z_{-K})| \cdots |\psi(z_{n-1})| \frac{z_n^m}{|\lambda|^n} \frac{b^m C}{|\lambda| - b^m C}.
\]

If, in addition we choose $m$ so that
\[
M |\psi(z_{-K})| \cdots |\psi(z_{n-1})| \frac{1}{|\lambda|^n} \frac{1}{\tilde{v}(z_n)} \frac{b^m C}{|\lambda| - b^m C} < |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{1}{2 \tilde{v}(z_0)},
\]
then
\[
\left| \sum_{k=n+1}^{\infty} \lambda^{-k} \psi(z_{-K}) \cdots \psi(z_{k-1}) g(z_k) \right| \\
\leq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{z_0^m}{2 \tilde{v}(z_0)} \\
\leq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{z_0^m}{2 \tilde{v}(z_0)}.
\]

Hence by (3.3)
\[
|L_{\xi,\psi}(g)| \geq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{z_0^m}{2 \tilde{v}(z_0)} - \left| \sum_{k=n+1}^{\infty} \lambda^{-k} \psi(z_{-K}) \cdots \psi(z_{k-1}) g(z_k) \right| \\
\geq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{z_0^m}{2 \tilde{v}(z_0)}.
\]
(4.4)

Therefore using Proposition 3.1, we obtain the desired lower bound
\[
\|L_{\xi,\psi}\|_{v,m} \geq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{z_0^m}{2M \tilde{v}(z_0)} \\
\geq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{1}{2Mm} \|\delta_{z_0}\|_{v,m}.
\]
(3.5)

The final step is to estimate $\|(C^*_m - \lambda I)L_{\xi,\psi}\|_{v,m}$ for a suitable iteration sequence $\xi$. First observe that
\[
(C^*_m - \lambda I)L_{\xi,\psi} = -\lambda^{K+1} \delta_{z_{-K}}.
\]
Recall that
\[
\rho = r_{e,H_0}(C_{\phi,\psi}) = \lim_n \| (C_{\phi,\psi})^n \|_{v,v}^{\frac{1}{n}}
\]
and
\[
(C_{\phi,\psi})^n f(w) = \psi(w) \cdots \psi(\phi_{n-1}(w)) C_{\phi_n} f(w), \quad w \in D, f \in H_0^\infty,
\]
where \( \varphi_n = \varphi \circ \cdots \circ \varphi \) (\( n \) times). Hence \((C_{\varphi,\psi})^n : H_v^\infty \to H_v^\infty\) is a bounded weighted composition operator and we get by Theorem 4.2 in [6] that

\[
\|(C_{\varphi,\psi})^n\|_{e,v} = \lim_{r \to 1^-} \sup_{|w| > r} |\psi(w)| \cdots |\psi(\varphi_{n-1}(w))| \frac{v(w)}{\tilde{v}(\varphi_n(w))}.
\]

Pick \( \mu \) so that \( |\lambda| < \mu < \rho \). Thus there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \),

\[
\|(C_{\varphi,\psi})^n\|_{e,v} > \mu^n > 0.
\]

Hence for any \( l \geq n_0 \) we can find a \( w \in D \) so that

\[
|\psi(w)| \cdots |\psi(\varphi_{l-1}(w))| \frac{v(w)}{\tilde{v}(\varphi_l(w))} \geq \mu^l > 0 \text{ and } |\varphi_l(w)| \geq \frac{1}{2}.
\]

This means that for every \( K \geq n_0 \) with the above choice of \( w \in D \) we can form an iteration sequence \((z_k)_{k=-K}^{\infty}\) by letting \( z_{-K} = w \) and \( z_{k+1} = \varphi(z_k) \) for \( k \geq -K \). Then \( |z_0| = |\varphi_K(w)| \geq 1/2 \). By (3.4) and (3.5) we obtain

\[
\|L_{\xi,\psi}\|_{v,m} \geq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{1}{2MMm} |z_0|^m.\]

Moreover

\[
\|\delta_{z_{-K}}\|_{v,m} \leq \|\delta_{z_{-K}}\|_v = \frac{1}{\tilde{v}(z_{-K})} \leq \frac{1}{v(z_{-K})}.
\]

Finally,

\[
\frac{\|(C_m^* - \lambda I)L_{\xi,\psi}\|_{v,m}}{\|L_{\xi,\psi}\|_{v,m}} \leq 2 \frac{MMm|\lambda|^{K+1}\tilde{v}(z_0)}{|\psi(z_{-K})| \cdots |\psi(z_{-1})||z_0|^m v(z_{-K})} \leq |\lambda| MMm2^{m+1} \left( \frac{|\lambda|}{\mu} \right)^K.
\]

By choosing \( K \geq n_0 \) large enough we see that \((C_m^* - \lambda I)\) is not bounded from below.

The proof of the following corollary is in fact the same proof as the proof given in [1], Corollary 8. For the sake of completeness we repeat it here.

**Corollary 3.1.** Let \( v \) be a typical weight. Suppose that \( \varphi \), not an automorphism, has fixed point \( a \in D \) and that \( C_{\varphi,\psi} : H_v^0 \to H_v^0 \) is bounded. Then

\[
\sigma_{H_v^0}(C_{\varphi,\psi}) = \{ \lambda \in \mathbb{C}; \ |\lambda| \leq r_{e,H_v^0}(C_{\varphi,\psi}) \} \cup \{ \psi(a)\varphi'(a) \}_{n=0}^\infty.
\]

**Proof.** Since \( C_{\varphi,\psi} : H_v^0 \to H_v^0 \) is bounded, we get that \( C_{\varphi,\psi}^* : H_v^\infty \to H_v^\infty \) is also bounded, and we can apply the previous theorem. Moreover \( C_{\varphi,\psi}^* \) is bounded on \( H_v^\infty \) and \( H_v^0 \) and and \( C_{\varphi,\psi}^n f(w) = \psi(w) \cdots \psi(\varphi_{n-1}(w))C_{\varphi,n}(f(w)) \). Hence \( C_{\varphi,\psi}^n \) is a weighted bounded weighted composition operator on \( H_v^0 \) and \( H_v^\infty \) and we can apply Theorem 2 in [13] to get that the essential norm of \( C_{\varphi,\psi}^n \) on \( H_v^0 \) coincides with the essential norm of \( C_{\varphi,\psi}^n \) on \( H_v^\infty \). Thus, \( r_{e,H_v^\infty}(C_{\varphi,\psi}) = r_{e,H_v^0}(C_{\varphi,\psi}) \), and the claim follows.  

\[\blacksquare\]
References


