The Edge Steiner Number of a Graph

MICHAEL B. FRONDOZA AND SERGIO R. CANOY, JR.
Department of Mathematics, College of Science and Mathematics,
MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines
michael.frondoza@g.msuiit.edu.ph, serge_canoy@yahoo.com

Abstract. The concepts of edge Steiner set and edge Steiner number of a graph are investigated in this study. A necessary and sufficient condition for a graph \( G \) to satisfy \( \text{st}_e(G) = |V(G)| - 1 \), where \( \text{st}_e(G) \) denotes the edge Steiner number of \( G \), is obtained. Edge Steiner sets in the joins of graphs are also studied and the Steiner numbers of these graphs are determined.

2010 Mathematics Subject Classification: 05C12

Keywords and phrases: Steiner \( W \)-tree, edge Steiner set, edge Steiner number.

1. Introduction

Given a connected graph \( G \) and a nonempty subset \( W \) of \( V(G) \), a Steiner \( W \)-tree is a tree of minimum order that contains \( W \). The sets \( S(W) \) and \( S_e(W) \) denote, respectively, the sets of all vertices and edges of \( G \) that lie on any Steiner \( W \)-tree. \( W \) is called a vertex Steiner set if \( S(W) = V(G) \). If \( S_e(W) = E(G) \), then \( W \) is said to be an edge Steiner set of \( G \). A vertex (edge) Steiner set of minimum cardinality is called a minimum vertex (edge) Steiner set. The cardinality of a minimum vertex (edge) Steiner set is defined as the vertex (edge) Steiner number \( \text{st}(G) \) (resp. \( \text{st}_e(G) \)) of \( G \).

Steiner sets and Steiner numbers have been studied recently in [1, 2, 4]. In [2], the authors characterized the Steiner sets in the join \( G + H \) and the composition \( G[H] \) of two nontrivial connected graphs \( G \) and \( H \). Edge Steiner sets, edge Steiner number, minimal edge Steiner sets, and upper edge Steiner numbers have been extensively studied very recently in [5]. For other terminologies, one may refer to [3].

2. Results

The following remarks are immediate from the definitions of edge Steiner set and edge Steiner number of a graph. The first and the third of these can be found in [5].

Remark 2.1. If \( G \) is a connected graph of order \( n \geq 2 \), then \( 2 \leq \text{st}_e(G) \leq n \).

Communicated by Lee See Keong.
Received: July 11, 2008; Revised: March 29, 2010.
**Remark 2.2.** Let $G$ be a connected graph of order $n \geq 2$. Then $\text{st}_e(G) < n$ if and only if there exists a proper subset $W$ of $V(G)$ such that $\langle W \rangle$ is disconnected and $S_e(W) = E(G)$.

**Remark 2.3.** $\text{st}_e(K_n) = n$ for each positive integer $n \geq 2$.

Next, we briefly define the concepts of independent cutset and essential independent cutset in a graph and look at some relationships between these concepts and the concept of edge Steiner set.

**Definition 2.1.** Let $G$ be a connected graph. A subset $Y$ of $V(G)$ is said to be an independent cutset (or simply an ics) in $G$ if it is independent and $(V(G) \setminus Y)$ is disconnected. $Y$ is said to be an essential independent cutset (or eics) if it is an ics and $\langle (V(G) \setminus Y) \cup \{y\} \rangle$ is connected for every $y \in Y$. An eics of $G$ of maximum cardinality is called a maximum eics of $G$.

**Example 2.1.** Consider the graph below.

\[
\begin{align*}
\end{align*}
\]

$S = \{v_2, v_4, v_6\}$ and $R = \{v_4, v_2\}$ are independent cutsets. $S$ is not an eics since $\langle V(G) \setminus S \rangle \cup \{v_6\}$ is disconnected. The set $R$ is an eics since $\langle V(G) \setminus R \rangle \cup \{v_2\}$ and $\langle V(G) \setminus R \rangle \cup \{v_4\}$ are connected.

**Example 2.2.** Consider another graph below.

\[
\begin{align*}
\end{align*}
\]

It can be verified that the sets $\{v_4, v_5\}$, $\{v_4, v_1\}$, $\{v_2, v_3\}$, $\{v_2, v_6\}$, $\{v_3, v_6\}$, $\{v_2, v_3, v_6\}$ are the only essential independent cutsets of $G$. Thus $U = \{v_2, v_3, v_6\}$ is a maximum eics of $G$.

**Remark 2.4.** A connected non-complete graph may have no eics.

To see this, consider the graph below.

\[
\begin{align*}
\end{align*}
\]
It can be verified that $G$ has no eics.

**Theorem 2.1.** Let $G$ be a connected non-complete graph of order $n \geq 2$. If $Y$ is an essential independent cutset of $G$, then $V(G) \setminus Y$ is an edge Steiner set of $G$.

**Proof.** Let $W = V(G) \setminus Y$, where $Y$ is an eics. Then $W$ is disconnected. Let $y \in Y$. Since $Y$ is an eics, $\langle W \cup \{y\} \rangle$ is connected. Hence every spanning tree of $\langle W \cup \{y\} \rangle$ is a Steiner $W$-tree. This implies that $E(\langle W \cup \{y\} \rangle) \subseteq S_e(W)$. Since $Y$ is independent, it follows that $E(G) = \cup_{y \in Y} E(\langle W \cup \{y\} \rangle) \subseteq S_e(W)$. Thus $W$ is an edge Steiner set of $G$.

The following result is immediate from Theorem 2.1.

**Corollary 2.1.** Let $G$ be a connected non-complete graph of order $n \geq 2$. If $G$ has an essential independent cutset, then $st_e(G) \leq n - r$, where $r = \max\{|Y| : Y \text{ is an eics in } G\}$.

**Remark 2.5.** The converse of Theorem 2.1 is not true.

To see this, consider again the graph in Example 2.4. The graph $G$ has no eics and $W = \{v_1, v_3, v_5\}$ is a minimum edge Steiner set of $G$. Thus $st_e(G) = 3 \neq 6$.

**Lemma 2.1.** Let $G$ be a connected graph and $v$ a cut-vertex of $G$. If $W \subseteq V(G)$ and $W \cap H \neq \emptyset$ for every component $H$ of $\langle V(G) \setminus \{v\} \rangle$, then $v \in V(T)$ for every Steiner $W$-tree $T$ of $G$.

**Proof.** Let $v$ be a cut-vertex of a connected graph $G$ and $W \subseteq V(G)$. Then $\langle V(G) \setminus \{v\} \rangle$ is disconnected. If $v \in W$, then we are done. Suppose that $v \notin W$. Let $Y_1, Y_2, \ldots, Y_k$ be the components of $\langle V(G) \setminus \{v\} \rangle$ and suppose that $V(Y_j) \cap W \neq \emptyset$ for all $j \in I = \{1, 2, \ldots, k\}$. Clearly, $\cup_{j \in I} (V(Y_j) \cap W) = W$; hence $\langle W \rangle = \langle \cup_{j \in I} (V(Y_j) \cap W) \rangle$ is disconnected. Now, let $T$ be a Steiner $W$-tree of $G$. Pick $v_1 \in V(Y_1) \cap W$ and $v_2 \in V(Y_2) \cap W$. Since $W \subseteq V(T)$, it follows that $v_1, v_2 \in V(T)$. Hence there is a path in $T$ connecting $v_1$ and $v_2$. Clearly, this path contains $v$. Therefore, $v \in V(T)$.

The next result is found in [5].

**Lemma 2.2.** Let $G$ be a connected graph and $v$ a cut-vertex of $G$. If $W$ is an edge Steiner set of $G$, then $v \in V(T)$ for every Steiner $W$-tree $T$ of $G$.

**Theorem 2.2.** Let $v$ be a cut-vertex of a connected graph $G$ and $W \subseteq V(G)$ with $v \notin W$. Then $W \cup \{v\}$ is an edge Steiner set of $G$ if and only if $W$ is an edge Steiner set of $G$.

**Proof.** Suppose that $W' = W \cup \{v\}$ is an edge Steiner set of $G$ and $e \in E(G)$. Since $S_e(W') = E(G)$, there exists a Steiner $W'$-tree $T_e$ of $G$ such that $e \in E(T_e)$. Since $W' \cap V(H) \neq \emptyset$ for every component $H$ of $\langle V(G) \setminus \{v\} \rangle$, $W \cap V(H) \neq \emptyset$ for every component $H$ of $\langle V(G) \setminus \{v\} \rangle$. By Lemma 2.1, $T_e$ is also a Steiner $W$-tree of $G$. Thus $e \in S_e(W)$, that is, $E(G) \subseteq S_e(W)$. Hence $E(G) = S_e(W)$. This implies that $W$ is also an edge Steiner set of $G$.

Conversely, assume that $W$ is an edge Steiner set of $G$ and let $e \in E(G)$. Since $S_e(W) = E(G)$ it follows that there exists a Steiner $W$-tree $T_e$ such that $e \in E(T_e)$.
Corollary 2.2. Let \( G \) be a connected graph and \( v \) a cut-vertex of \( G \). If \( W \) is a minimum edge Steiner set of \( G \), then \( v \notin W \).

The next three results are also quick consequences of Theorem 2.2.

Corollary 2.3. Let \( G \) be a connected graph of order \( n \) and \( W \) an edge Steiner set of \( G \). If \( C \) is the set of cut-vertices of \( G \), then \( W \cap C \) is an edge Steiner set of \( G \).

Proof. Let \( C = \{v_1, v_2, \ldots, v_k\} \). Clearly, \( W \setminus C = W \setminus (W \cap C) \). If \( C \cap W = \emptyset \), then \( W \setminus C = W \). Hence \( W \setminus C \) is an edge Steiner set of \( G \). Assume that \( C \cap W \neq \emptyset \), say \( |C_o| = \{y_1, y_2, \ldots, y_m\} \). Since \( W \) is an edge Steiner set of \( G \), \( Y_1 = W \setminus \{y_1\} \) is also an edge Steiner set of \( G \) by Theorem 2.2. Again, by Theorem 2.2, \( Y_2 = Y_1 \setminus \{y_2\} \) is an edge Steiner set of \( G \). Repeating the process for the remaining vertices of \( C_o \), it follows that \( Y_m = Y_{m-1} \setminus \{y_m\} \) is an edge Steiner set of \( G \). Therefore \( Y_m = Y_1 \setminus \{y_2, y_3, \ldots, y_{m-1}, y_m\} = W \setminus C_o = W \setminus C \) is an edge Steiner set of \( G \).

Corollary 2.4. Let \( G \) be a connected graph and \( C \) the set containing all the cut-vertices of \( G \). Then any superset \( W_o \) of \( V(G) \setminus C \) is an edge Steiner set of \( G \).

Proof. Let \( C_o = W_o \cap C \). If \( C_o = \emptyset \), then \( W_o = V(G) \setminus C \) is an edge Steiner set by Corollary 2.3. So, suppose \( C_o \neq \emptyset \), say \( C_o = \{x_1, x_2, \ldots, x_m\} \). Since \( x_1 \notin V(G) \setminus C \), it follows from Theorem 2.2 that \( Y_1 = (V(G) \setminus C) \cup \{x_1\} \) is also an edge Steiner set of \( G \). Again, since \( x_2 \notin Y_1 \), \( Y_2 = Y_1 \cup \{x_2\} \) is an edge Steiner set of \( G \). Proceeding in this manner, we find that \( W_o = Y_m = Y_{m-1} \cup \{x_m\} \) is an edge Steiner set of \( G \).

Corollary 2.5. If \( G \) is a connected graph and \( q \) is the number of cut-vertices of \( G \), then \( st_e(G) \leq |V(G)| - q \).

Proof. Let \( C = \{v : v \text{ is a cut-vertex of } G\} \). From Corollary 2.3 and the fact that \( V(G) \) is an edge Steiner set of \( G \), it follows that \( V(G) \setminus C \) is an edge Steiner set of \( G \). Hence, if \( |C| = q \), then \( st_e(G) \leq |V(G) \setminus C| = |V(G)| - |C| = |V(G)| - q \).

Theorem 2.3. Let \( G \) be a connected graph of order \( n \geq 2 \). Then \( st_e(G) = n - 1 \) if and only if \( G \) has a unique cut-vertex \( v \) such that \( st_e(\langle V(H) \cup \{v\} \rangle) = |V(H)| + 1 \) for every component \( H \) of \( \langle V(G) \setminus \{v\} \rangle \).

Proof. Let \( G \) be a connected graph of order \( n \) and \( st_e(G) = n - 1 \). Then there exists a vertex \( v \in V(G) \) such that \( W = V(G) \setminus \{v\} \) is an edge Steiner set of \( G \). Since \( \langle W \rangle \) is disconnected, \( v \) is a cut-vertex of \( G \). From Corollary 2.5, \( v \) is the unique cut-vertex of \( G \). Let \( Y_1, Y_2, \ldots, Y_k \) be the components of \( \langle V(G) \setminus \{v\} \rangle \). Suppose that \( st_e(\langle V(Y_m) \cup \{v\} \rangle) < |V(Y_m)| + 1 \) for some \( m \), where \( 1 \leq m \leq k \). Let \( W_{Y_m} \) be a minimum edge Steiner set of \( \langle V(Y_m) \cup \{v\} \rangle \). Then \( \langle W_{Y_m} \rangle \) is a disconnected proper subgraph of \( \langle V(Y_m) \cup \{v\} \rangle \). Let \( W_o = \bigcup_{i \neq m} V(Y_i) \) and let \( W^* = (W_o \cup W_{Y_m}) \). Since \( v \) is a cut-vertex of \( \langle (\bigcup_{i \neq m} V(Y_i)) \cup \{v\} \rangle \), it follows that \( (\bigcup_{i \neq m} V(Y_i)) \cup \{v\} = \bigcup_{i \neq m} V(Y_i) \) is an edge Steiner set of \( \langle (\bigcup_{i \neq m} V(Y_i)) \cup \{v\} \rangle \) by Theorem 2.2. Let \( A = W_o \cup \{v\} \), \( B = V(Y_m) \cup \{v\} \) and \( e \in E(G) \). Consider the
following cases.

**Case 1:** \( e \in E(A) \).

Since \( W_o \) is an edge Steiner set of \( A \), there exists a Steiner \( W_o \)-tree \( T_e \) of \( A \) such that \( e \in E(T_e) \). Choose \( u \in V(Y_m) \) such that \( e' = uv \in E(B) \). Since \( W_{Y_m} \) is an edge Steiner set of \( B \), there exists a Steiner \( W_{Y_m} \)-tree \( T'_e \) of \( B \) such \( e' \in E(T'_e) \). Clearly, \( v \in V(T_e) \cap V(T'_e) \). Let \( T(e) \) be the tree obtained by gluing \( T_e \) and \( T'_e \) at vertex \( v \). Then \( T(e) \) is a Steiner \( W^* \)-tree of \( G \) with \( e \in E(T(e)) \).

**Case 2:** \( e \in E(B) \).

Let \( T \) be a Steiner \( W_o \)-tree of \( A \). Since \( W_{Y_m} \) is an edge Steiner set, there exists a Steiner \( W_{Y_m} \)-tree \( T_e \) with \( e \in E(T_e) \). Consider the following subcases:

**Subcase 1:** \( v \in W_{Y_m} \).

Then \( v \in V(T_e) \). Let \( T(e) \) be the tree obtained by gluing \( T_e \) and \( T \) at the vertex \( v \). Then \( T(e) \) is a Steiner \( W^* \)-tree of \( G \) with \( e \in E(T(e)) \).

**Subcase 2:** \( v \notin W_{Y_m} \).

Extend (if necessary) \( T_e \) to a tree \( T_{uv} \) \((u \in V(Y_m))\) of minimum order such that \( v \in V(T_{uv}) \). Let \( T(e) \) be the tree obtained by gluing \( T_{uv} \) and \( T \) at the vertex \( v \). Then \( T(e) \) is a Steiner \( W^* \)-tree of \( G \) with \( e \in E(T(e)) \).

In any case, \( S_e(W^*) = E(G) \). Consequently, \( W^* \) is an edge Steiner set of \( G \). By Corollary 2.3, \( W^* \backslash \{v\} \) is also an edge Steiner set of \( G \). If \( v \in W_{Y_m} \), then \( v \in W^* \) and \( n - 1 = st_e(G) \leq |W^* \backslash \{v\}| = |W^*| - 1 = |W_o| + |W_{Y_m}| - 1 < |W_o| + |V(Y_m)| + 1 - 1 = n - 1 \), which is a contradiction. If \( v \notin W_{Y_m} \), then \( \langle Y_{Y_m} \rangle \) is a disconnected subgraph of \( \langle Y_{Y_m} \rangle \). Thus \( W_{Y_m} \leq |V(Y_m)| - 1 \) and \( st_e(G) \leq |W^* \backslash \{v\}| = |W^*| \leq n - 2 \), contrary to the assumption that \( st_e(G) = n - 1 \). Therefore, \( st_e((V(H) \cup \{v\})) = |V(H)| + 1 \) for every component \( H \) of \( G \) \( \backslash \{v\} \).

Conversely, assume that there exists a unique cut-vertex \( v \) such that for every component \( H \) of \( G \backslash u \), \( st_e((H \cup \{v\})) = |V(H)| + 1 \). Then by Corollary 2.5, \( st_e(G) \leq n - 1 \). Suppose that \( st_e(G) < n - 1 \). Then there exists \( W^* \subset V(G) \) such that \( S_e(W^*) = E(G) \) and \( st_e(G) = |W^*| < |V(G)| - 1 \). By Corollary 2.2, \( v \notin W^* \). This implies that there exists a component \( H \) of \( G \backslash v \) such that \( V(H) \cap W^* \subset V(H) \). Let \( W_H = V(H) \cap W^* \). Let \( e \in E((V(H) \cup \{v\})) \). Then \( e \in E(G) \) and \( e \in E(T_e) \) for some Steiner \( W^* \)-tree \( T_e \) of \( G \). Let \( T_e \) be the portion of the tree \( T_e \), where \( V(T_e) = V(T_e) \cap (V(H) \cup \{v\}) \). Then \( T_e \) is a Steiner \((W_H \cup \{v\})\)-tree of \((V(H) \cup \{v\}) \) and \( e \in E(T_e) \). Hence \( W_H \cup \{v\} \) is an edge Steiner set of \((V(H) \cup \{v\}) \). This implies that \( st_e((V(H) \cup \{v\})) \leq |W_H \cup \{v\}| < |V(H)| + 1 \), contrary to the assumption.

The next result characterizes the edge Steiner sets in a join of two graphs.

**Theorem 2.4.** Let \( G \) and \( H \) be graphs of orders \( n \) and \( m \), respectively, such that none of them is the empty graph. Then \( W \subseteq V(G + H) \) is an edge Steiner set of \( G \) if and only if \( W = V(G + H) \).

**Proof.** Suppose that \( W \) is an edge Steiner set of \( G + H \). Let \( W_1 = W \cap V(G) \) and \( W_2 = W \cap V(H) \). If \( W_1 = \emptyset \), then \( W = W_2 \subseteq V(H) \). Since \( W \) is a Steiner set of \( V(G + H) \), the graph \( \langle W \rangle \) induced by \( W \) must be disconnected. Let \( v \in V(G) \). Then
\[ W \cup \{v\} \] is a connected subgraph of \( G+H \). This implies that every Steiner \( W \)-tree of \( G+H \) has exactly \(|W|+1\) vertices. Since \( G \) is not an empty graph, there exist \( x, y \in V(G) \) such that \( xy \in E(G+H) \). Clearly, this edge cannot be in any Steiner \( W \)-tree of \( G+H \). This contradicts our assumption that \( W \) is an edge Steiner set of \( G+H \). Therefore \( W_1 \neq \emptyset \). Similarly, \( W_2 \neq \emptyset \). Consequently, \( \langle W \rangle \) is a connected subgraph of \( G+H \) and so any Steiner \( W \)-tree of \( G+H \), therefore, has \(|W|\) vertices. Since \( W \) is an edge Steiner set of \( G+H \), it follows that \( W = V(G+H) \).

The converse is clear.

An immediate consequence of the Theorem 2.4 is the following result.

**Corollary 2.6.** Let \( G \) and \( H \) be graphs of orders \( n \) and \( m \), respectively, such that none of them is the empty graph. Then \( st_e(G+H) = n + m \).

**Theorem 2.5.** Let \( G \) and \( H \) be graphs of orders \( n \) and \( m \), respectively, such that \( G+H \) is not a star, and at least one of them is the empty graph. Then \( W \subseteq V(G+H) \) is an edge Steiner set of \( G \) if and only if either

(i) \( W = V(G+H) \);

(ii) \( W = V(G) \), \( G \) is disconnected, and \( H = \overline{K}_m \); or

(iii) \( W = V(H) \), \( H \) is disconnected, and \( G = \overline{K}_n \).

**Proof.** Suppose that \( W \) is an edge Steiner set of \( G+H \). Suppose \( W \neq V(G+H) \). Then \( \langle W \rangle \) is disconnected and so \( W \subseteq V(G) \) or \( W \subseteq V(H) \). Furthermore, any Steiner \( W \)-tree of \( G+H \) will have \(|W|+1\) vertices. Assume that \( W \subseteq V(G) \) and suppose that \( W \neq V(G) \). Pick \( v \in V(G) \setminus W \) and \( u \in V(H) \). Then none of the Steiner \( W \)-trees of \( G+H \) can contain \( uv \in E(G+H) \), contrary to our assumption of \( W \). Thus \( W = V(G) \). In this case, \( H = \overline{K}_m \); otherwise, there exist \( a, b \in V(H) \) such that \( ab \in E(G+H) \). However, none of the possible Steiner \( W \)-trees can contain the edge \( ab \), contradicting again our assumption. Similarly, if \( W \subseteq V(H) \), then \( W = V(H) \), \( H \) is disconnected, and \( G = \overline{K}_n \).

The converse can easily be proved.

The following result is a direct consequence of Theorem 2.5.

**Corollary 2.7.** Let \( G \) and \( H \) be graphs of orders \( n \) and \( m \), respectively, such that \( G+H \) is not a star, and at least one of them is the empty graph. Then

\[
st_e(G+H) = \begin{cases} 
  n, & \text{if } G \text{ is disconnected, } G \neq \overline{K}_n, \text{ and } H = \overline{K}_m \\
  m, & \text{if } H \text{ is disconnected, } H \neq \overline{K}_m, \text{ and } G = \overline{K}_n \\
  \min\{n, m\}, & \text{if } G = K_{m,n} \\
  n + m, & \text{otherwise.} 
\end{cases}
\]

**Corollary 2.8.** Let \( n \) and \( m \) be positive integers.

(a) \( st_e(K_n + P_m) = n + m \) \( (m \geq 2) \)

(b) \( st_e(K_n + C_m) = n + m \) \( (m \geq 3) \)

(c) \( st_e(K_{n_1, n_2, \ldots, n_k}) = \sum_{i=1}^{k} n_i \), where \( k \geq 3 \).
Acknowledgement. The authors are very grateful to the referee for pointing out errors in the initial manuscript and for giving helpful comments which led to the improvement of this paper.

References


