On the Definition of Atanassov’s Intuitionistic Fuzzy Subrings and Ideals

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Abstract. On the basis of the concept of grades of a fuzzy point to belongingness (∈) or quasi-coincident (q) or belongingness and quasi-coincident (∈ ∧ q) or belongingness or quasi-coincident (∈ ∨ q) in an intuitionistic fuzzy set of a ring, the notion of a (α, β)-intuitionistic fuzzy subring and ideal is introduced by applying the Lukasiewicz 3-valued implication operator. Using the notion of fuzzy cut set of an intuitionistic fuzzy set, the support and α-level set of an intuitionistic fuzzy set are defined and it is established that, for α ≠ ∈ ∧ q, the support of a (α, β)-intuitionistic fuzzy ideal of a ring is an ideal of the ring. It is also established that the level sets of an intuitionistic fuzzy ideal with thresholds (s, t) of a ring is an ideal of the ring. We investigate that an intuitionistic fuzzy set A of a ring is a (∈, ∈) (or (∈, ∈ ∨ q) or (∈ ∧ q, ∈))-intuitionistic fuzzy ideal of the ring if and only if A is an intuitionistic fuzzy ideal with thresholds (0, 1) (or (0, 0.5) or (0.5, 1)) of the ring respectively. We also establish that for any a ∈ (s, t], the cut set A_a is a fuzzy ideal of the ring. Finally, we investigate that an intuitionistic fuzzy set of a ring is an intuitionistic fuzzy ideal with thresholds (s, t) of the ring if and only if for any a ∈ (s, t], the cut set A_a is a fuzzy ideal of R.

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1. Introduction

Since the introduction of fuzzy sets by Zadeh [26] in 1965, the researchers have been carrying out research in various concepts of abstract algebra in fuzzy setting. Fuzzy subgroups of a group was introduced by Rosenfeld [19] in 1971. Since then many generalization of this fundamental concept have been done. A self contained survey of the state of art of the fuzzy binary relations and some of their applications has been provided by Beg and Ashraf in [4]. Bhakat and Das in [5, 6], redefined fuzzy subgroups of a group using the notion of belongings to (∈) and quasi-coincident (q) of a fuzzy point to a fuzzy set of the
group. In [7], fuzzy subring and ideal are redefined. Davvaz et al. in [9,10], generalized the concept to \(H_t\)-submodules and redefined fuzzy \(H_t\)-submodules by applying many valued implication operators. In [14] the notion of interval valued fuzzy \(k\)-ideals of semirings is introduced, which is a generalization of a fuzzy \(k\)-ideal. As a generalization of fuzzy set, intuitionistic fuzzy set was introduced by Atanassov [1], also see [2,3]. Since then various concepts of fuzzy setting have been generalized to intuitionistic fuzzy set, for example see [8,11–13,15,24]. Fuzzy aspects of ordered semigroups have been studied by many researchers as seen in [16,20,21]. Characterization of different types of \((\alpha, \beta)\)-intuitionistic fuzzy subgroups \(A\) of a group using the notions of grades of a fuzzy point belongs to \(A\) or quasi-coincident with \(A\) or belongs to and quasi-coincident \((\in \land q)\) or belongs to or quasi-coincident \((\in \lor q)\) has been done in [23]. Intuitionistic fuzzy ideal with thresholds \((s,t)\) of a ring was introduced in [22]. In this paper, using the notions of grades of a fuzzy point \(x_a\) belongs to an intuitionistic fuzzy set \(A\), in a ring \(R\) or quasi-coincident with \(A\) or belongs to and quasi-coincident \((\in \land q)\) or belongs to or quasi-coincident \((\in \lor q)\), a \((\alpha, \beta)\)-intuitionistic fuzzy subring and ideal is defined by applying the Lukasiewicz 3-valued implication operator, see [17]. The support and \(\alpha\)-level set of an intuitionistic fuzzy set is defined based on fuzzy cut set and grades of belongs to respectively. It is established that, for \(\alpha \neq \in \land q\), the support of a \((\alpha, \beta)\)-intuitionistic fuzzy ideal of a ring is an ideal of the ring. We investigate that the level sets of an intuitionistic fuzzy ideal with thresholds \((s,t)\) of a ring is an ideal of the ring. We obtain necessary and sufficient conditions between \((\alpha, \beta)\)-intuitionistic fuzzy ideal and intuitionistic fuzzy ideal with thresholds \((s,t)\). It is established that an intuitionistic fuzzy set \(A\) of a ring is a \((\in, \in)\) or \((\in, \in \land q)\) or \((\in \land q, \in)\) or \((\in \land q, \in)\)-intuitionistic fuzzy ideal of the ring if and only if \(A\) is an intuitionistic fuzzy ideal with thresholds \((0,1)\) or \((0,0.5)\) or \((0.5,1)\) of the ring respectively. We also establish that \(A\) is a \((\in, \in)\) or \((\in, \in \land q)\) or \((\in \land q, \in)\)-intuitionistic fuzzy ideal of the ring if and only if for any \(a \in \{0,1\}\) or \(a \in \{0,0.5\}\) or \(a \in \{0.5,1\}\), \(A_a\) is a fuzzy ideal of the ring respectively. Finally, we investigate that an intuitionistic fuzzy set of a ring is an intuitionistic fuzzy ideal with thresholds \((s,t)\) of the ring if and only if for any \(a \in \{s,t\}\), the cut set \(A_a\) is a fuzzy ideal of \(R\).

### 2. Basic definitions and notations

A ring is a non-empty set \(R\) having two binary operations addition (+) and multiplication (·), where \((R,+)\) is a commutative group, \((R,\cdot)\) is a semigroup and addition is distributive with respect to multiplication. By zero \((0)\) we mean the additive identity of \(R\). A non-empty subset \(I\) of \(R\) is called an ideal of \(R\), if for any \(x, y \in I\) and \(r \in R\), we have \(x \cdot y, r x, x r \in I\).

A fuzzy set on a non-empty set was introduced by Zadeh [26] in 1965 and was defined as follows:

By a fuzzy set of a ring \(R\), we mean any mapping \(\mu\) from \(R\) to \([0,1]\). By \([0,1]^{R}\) we will denote the set of all fuzzy subsets of \(R\). For each fuzzy set \(\mu\) in \(R\) and any \(\alpha \in \{0,1\}\), we define two sets

\[
U(\mu, \alpha) = \{x \in R \mid \mu(x) \geq \alpha\} \quad \text{and} \quad L(\mu, \alpha) = \{x \in R \mid \mu(x) \leq \alpha\},
\]

which are called an upper level cut and a lower level cut of \(\mu\), respectively. The complement of \(\mu\), denoted by \(\mu^c\), is the fuzzy set on \(R\) defined by \(\mu^c(x) = 1 - \mu(x)\).

Let \(x \in R\) and \(t \in \{0,1\}\), then a fuzzy subset \(\mu \in [0,1]^R\) is called a fuzzy point if

\[
\mu(y) = \begin{cases} 
1, & \text{if } y = x \\
0, & \text{if } y \neq x 
\end{cases}
\]
and it is denoted by \( x_t \).

**Definition 2.1.** [5] Let \( \mu \) be a fuzzy subset of \( R \) and \( x_a \) be a fuzzy point. Then

1. If \( \mu(x) \geq a \), then we say \( x_a \) belongs to \( \mu \), and it is denoted by \( x_a \in \mu \).
2. If \( \mu(x) + a > 1 \), then we say \( x_a \) is quasi-coincident with \( \mu \), and it is denoted by \( x_aq\mu \).
3. If \( x_a \in \mu \), then we say \( x_a \) is quasi-coincident with \( \mu \), and it is denoted by \( x_aq\mu \).

The symbol \( \in \lor \) means that \( \in \lor \) does not hold. Let \( \mu, \sigma \in [0,1]^R \). Then, the intersection and union of \( \mu \) and \( \sigma \) are given by the fuzzy sets \( \mu \cap \sigma \) and \( \mu \cup \sigma \) respectively and are defined as follows:

1. \( (\mu \cap \sigma)(x) = \mu(x) \land \sigma(x) \);
2. \( (\mu \cup \sigma)(x) = \mu(x) \lor \sigma(x) \);

where \( \mu(x) \land \sigma(x) = \min\{\mu(x), \sigma(x)\} \) and \( \mu(x) \lor \sigma(x) = \max\{\mu(x), \sigma(x)\} \).

**Definition 2.2.** [18] Let \( R \) be a ring and \( \mu \) be a fuzzy subset in \( R \). Then, \( \mu \) is called a fuzzy subring of \( R \) if and only if for every \( x, y \in R \) the following conditions are satisfied:

1. \( \mu(x+y) \geq \mu(x) \land \mu(y) \);
2. \( \mu(-x) \geq \mu(x) \);
3. \( \mu(xy) \geq \mu(x) \lor \mu(y) \).

**Definition 2.3.** [18] Let \( R \) be a ring and \( \mu \) be a fuzzy subset in \( R \). Then, \( \mu \) is called a fuzzy ideal of \( R \) if and only if for every \( x, y \in R \) the following conditions are satisfied:

1. \( \mu(x+y) \geq \mu(x) \land \mu(y) \);
2. \( \mu(-x) \geq \mu(x) \);
3. \( \mu(xy) \geq \mu(x) \lor \mu(y) \).

An intuitionistic fuzzy set (abbreviated as IFS) introduced by Atanassov in [1] was defined as follows: An intuitionistic fuzzy set in a ring \( R \), is an object of the form \( A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in R\} \), where \( \mu_A \) and \( \nu_A \) are fuzzy sets in \( R \) and denote the degree of membership (namely \( \mu_A(x) \)) and the degree of non-membership (namely \( \nu_A(x) \)) of each element \( x \in R \) to the set \( A \) respectively, and \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \) for all \( x \in R \). By IFS\( (R) \) we denote the set of all IFSs of \( R \).

Let \( A=(\mu_A, \nu_A) \) and \( B=(\mu_B, \nu_B) \) be IFSs of \( R \). Then

1. \( A \subseteq B \) if and only if \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) for all \( x \in R \);
2. \( A \cup B = \{(x, \mu_A(x) \lor \mu_B(x)), \nu_A(x) \land \nu_B(x) \mid x \in R\} \);
3. \( A \cap B = \{(x, \mu_A(x) \land \mu_B(x)), \nu_A(x) \lor \nu_B(x) \mid x \in R\} \).

For our convenience we shall use the notation \( A(x) \geq B(x) \), when \( \mu_A(x) \geq \mu_B(x) \) and \( \nu_A(x) \leq \nu_B(x) \) for all \( x \in R \).

**Definition 2.4.** [22] Let \( A=(\mu_A, \nu_A) \) be an intuitionistic fuzzy set in \( R \). Then, \( A \) is said to be an intuitionistic fuzzy ideal with thresholds \( (\alpha, \beta) \) of \( R \), if it satisfies the following properties:

1. \( \mu_A(x+y) \lor \alpha \geq (\mu_A(x) \land \mu_A(y)) \land \beta \);
2. \( \mu_A(-x) \lor \alpha \geq \mu_A(x) \land \beta \);
3. \( \mu_A(xy) \lor \alpha \geq (\mu_A(x) \lor \mu_A(y)) \land \beta \);
4. \( \nu_A(x+y) \land (1-\alpha) \leq (\nu_A(x) \lor \nu_A(y)) \lor (1-\beta) \);
(5) \( v_A(-x) \land (1 - \alpha) \leq v_A(x) \lor (1 - \beta); \)
(6) \( v_A(xy) \land (1 - \alpha) \leq (v_A(x) \land v_A(y)) \lor (1 - \beta). \)
for all \( x, y \in R \), where \( \alpha, \beta \in [0, 1] \).

**Definition 2.5.** [25] Let \( A = (\mu_A, v_A) \) be an IFSs of \( R \), and \( a \in [0, 1] \). Then

(1)
\[
A_a(x) = \begin{cases} 
1, & \text{if } \mu_A(x) \geq a \\
\frac{1}{2}, & \text{if } \mu_A(x) < a \leq 1 - v_A(x) \\
0, & \text{for } a > 1 - v_A(x)
\end{cases}
\]

and
\[
A_{\bar{a}}(x) = \begin{cases} 
1, & \text{if } \mu_A(x) > a \\
\frac{1}{2}, & \text{if } \mu_A(x) \leq a < 1 - v_A(x) \\
0, & \text{for } a \geq 1 - v_A(x)
\end{cases}
\]
are called the \( a \)-upper cut set and \( a \)-strong upper cut set of \( A \), respectively.

(2)
\[
A^a(x) = \begin{cases} 
1, & \text{if } v_A(x) \geq a \\
\frac{1}{2}, & \text{if } v_A(x) < a \leq 1 - \mu_A(x) \\
0, & \text{for } a > 1 - \mu_A(x)
\end{cases}
\]

and
\[
A^{\bar{a}}(x) = \begin{cases} 
1, & \text{if } v_A(x) > a \\
\frac{1}{2}, & \text{if } v_A(x) \leq a < 1 - \mu_A(x) \\
0, & \text{for } a \geq 1 - \mu_A(x)
\end{cases}
\]
are called the \( a \)-lower cut set and \( a \)-strong lower cut set of \( A \), respectively.

(3)
\[
A_{[a]}(x) = \begin{cases} 
1, & \text{if } \mu_A(x) + a \geq 1 \\
\frac{1}{2}, & \text{if } \mu_A(x) \leq a < 1 - \mu_A(x) \\
0, & \text{for } a < v_A(x)
\end{cases}
\]

and
\[
A_{[\bar{a}]}(x) = \begin{cases} 
1, & \text{if } \mu_A(x) + a > 1 \\
\frac{1}{2}, & \text{if } \mu_A(x) < a \leq 1 - \mu_A(x) \\
0, & \text{for } a \leq v_A(x)
\end{cases}
\]
are called the \( a \)-upper \( Q \)-cut set and \( a \)-strong upper \( Q \)-cut set of \( A \), respectively.

(4)
\[
A^{[a]}(x) = \begin{cases} 
1, & \text{if } v_A(x) + a \geq 1 \\
\frac{1}{2}, & \text{if } v_A(x) \leq a < 1 - v_A(x) \\
0, & \text{for } a < v_A(x)
\end{cases}
\]
and
\[ A[a](x) = \begin{cases} 
1, & \text{if } \nu_A(x) + a > 1 \\
\frac{1}{2}, & \text{if } \mu_A(x) < a \leq 1 - \nu_A(x) \\
0, & \text{for } a \leq \mu_A(x)
\end{cases} \]
are called the a-lower Q-cut set and a-strong lower Q-cut set of A, respectively.

**Definition 2.6.** [23] Let \( A = (\mu_A, \nu_A) \) be an IFSs of R, and \( a \in [0, 1] \), \( x \in R \). Then

1. The grades of \( x_a \in A \) and \( x_a qA \) denoted by \([x_a \in A]\) and \([x_a qA]\) respectively are given by the following relations:
   \[ [x_a \in A] = A_a(x) \text{ and } [x_a qA] = A_a(x) \]

2. The grades of \( x_a \in qA \) and \( x_a \in qA \) denoted by \([x_a \in qA]\) and \([x_a \in qA]\) respectively are given by the following relations:
   \[ [x_a \in qA] = [x_a \in A] \quad \text{and} \quad [x_a qA] = A_a(x) \quad \text{and} \quad [x_a qA] = \]

3. The grades of \( x_a \in A \) and \( x_a \in qA \) denoted by \([x_a \in qA]\) and \([x_a \in qA]\) respectively are given by the following relations:
   \[ [x_a \in A] = A_a(x) \quad \text{and} \quad [x_a qA] = A_a(x) \quad \text{and} \quad [x_a qA] = \]

4. The grades of \( x_a \in qA \) and \( x_a \in qA \) denoted by \([x_a \in qA]\) and \([x_a \in qA]\) respectively are given by the following relations:
   \[ [x_a \in qA] = [x_a \in A] \quad \text{and} \quad [x_a qA] = A_a(x) \quad \text{and} \quad [x_a qA] = \]

Table 1. The table of truth value of Lukasiewicz implication.

<table>
<thead>
<tr>
<th>→</th>
<th>0</th>
<th>1/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
</tr>
</tbody>
</table>

As in [23] we have

1. \([x_a \in A] = [x_a \in A^c], [x_a qA] = [x_a qA^c] \).
2. \([x_a \in A] = [x_a \in A^c], [x_a \in qA] = [x_a \in qA^c] \).
3. \([x_a \in (\bigcap_{t \in T} A_t)] = \bigcap_{t \in T} [x_a \in A], [x_a qA(\bigcap_{t \in T} A_t)] = \bigcap_{t \in T} [x_a qA] \).
4. \([x_a \in (\bigcup_{t \in T} A_t)] = \bigcup_{t \in T} [x_a \in A], [x_a qA(\bigcup_{t \in T} A_t)] = \bigcup_{t \in T} [x_a qA] \).

In the next section we present our main results.

### 3. Main results

Let \( R \) be a ring and \( \alpha, \beta \in \{\in, q, \in \land q, \in \lor q\} \). Then, for \( a \in [0, 1] \), \( x \in R \), \( x_a \) is a fuzzy point and \([x_a \alpha A], [x_a \beta A] \in \{0, 1/2, 1\}\).

**Definition 3.1.** Let \( R \) be a ring and \( A = (\mu_A, \nu_A) \) be an IF set in \( R \). If for any \( \alpha, \beta \in \{\in, q, \in \land q, \in \lor q\} \), \( s, t \in (0, 1) \), and \( x, y \in R \), the following conditions are satisfied:
Consider $A$, then we can verify that

\[
\begin{align*}
(1) & \quad ([x_s \alpha A] \land [y_r \alpha A] \rightarrow [(x_s + y_r) \beta A]) = 1; \\
(2) & \quad ([x_s \alpha A] \rightarrow [-x_s \beta A]) = 1; \\
(3) & \quad ([x_s \alpha A] \land [y_r \alpha A] \rightarrow [(x_s y_r) \beta A]) = 1; \text{ then } A \text{ is called a } (\alpha, \beta)\text{-intuitionistic fuzzy subring of } R, \text{ where } (x_s + y_r) = (x + y)_{x \lor y}, -x_s = (-x)_s, \text{ and } (x_s y_r) = (xy)_{x \lor y}.
\end{align*}
\]

It is to note that, for $p, q \in \{0, 1/2, 1\}$, we have from Table 1, $(p \rightarrow q) = 1 \Leftrightarrow q \geq p$. Therefore, Definition 3.1 is equivalent to the following definition.

**Definition 3.2.** Let $R$ be a ring and $A = (\mu_A, \nu_A)$ be an IF set in $R$. If for any $\alpha, \beta \in \{\in, q, \in \land q, \in \lor q\}$, $s, t \in (0, 1]$, and $x, y \in R$, the following conditions are satisfied

\[
\begin{align*}
(1) & \quad ([x_s \alpha A] \land [y_r \alpha A] \rightarrow [(x_s + y_r) \beta A]) = 1; \\
(2) & \quad ([x_s \alpha A] \rightarrow [-x_s \beta A]) = 1; \\
(3) & \quad ([x_s \alpha A] \land [y_r \alpha A] \rightarrow [(x_s y_r) \beta A]) = 1; \text{ then } A \text{ is called a } (\alpha, \beta)\text{-intuitionistic fuzzy subring of } R, \text{ where } (x_s + y_r) = (x + y)_{x \lor y}, -x_s = (-x)_s, \text{ and } (x_s y_r) = (xy)_{x \lor y}.
\end{align*}
\]

**Definition 3.3.** Let $R$ be a ring and $A = (\mu_A, \nu_A)$ be an IF set in $R$. If for any $\alpha, \beta \in \{\in, q, \in \land q, \in \lor q\}$, $s, t \in (0, 1]$, and $x, y \in R$, the following conditions are satisfied

\[
\begin{align*}
(1) & \quad ([x_s \alpha A] \land [y_r \alpha A] \rightarrow [(x_s + y_r) \beta A]) = 1; \\
(2) & \quad ([x_s \alpha A] \rightarrow [-x_s \beta A]) = 1; \\
(3) & \quad ([x_s \alpha A] \lor [y_r \alpha A] \rightarrow [(x_s y_r) \beta A]) = 1; \text{ then } A \text{ is called a } (\alpha, \beta)\text{-intuitionistic fuzzy ideal of } R, \text{ where } (x_s + y_r) = (x + y)_{x \lor y}, -x_s = (-x)_s, \text{ and } (x_s y_r) = (xy)_{x \lor y}.
\end{align*}
\]

This is equivalent to:

**Definition 3.4.** Let $R$ be a ring and $A = (\mu_A, \nu_A)$ be an IF set in $R$. If for any $\alpha, \beta \in \{\in, q, \in \land q, \in \lor q\}$, $s, t \in (0, 1]$, and $x, y \in R$, the following conditions are satisfied:

\[
\begin{align*}
(1) & \quad ([x_s \alpha A] \land [y_r \alpha A] \rightarrow [(x_s + y_r) \beta A]) = 1; \\
(2) & \quad ([x_s \alpha A] \rightarrow [-x_s \beta A]) = 1; \\
(3) & \quad ([x_s \alpha A] \lor [y_r \alpha A] \rightarrow [(x_s y_r) \beta A]) = 1; \text{ then } A \text{ is called a } (\alpha, \beta)\text{-intuitionistic fuzzy ideal of } R, \text{ where } (x_s + y_r) = (x + y)_{x \lor y}, -x_s = (-x)_s, \text{ and } (x_s y_r) = (xy)_{x \lor y}.
\end{align*}
\]

**Example 3.1.** Consider the ring $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$, where operations are addition modulo 4 and multiplication modulo 4. Let $A = \{0, 2\}$. Then, $A$ is an ideal of $R$. We consider the following IFS of $R$

\[
\mu_A(x) = \begin{cases} 
0.4, & \text{if } x \in A \\
0.2, & \text{for } x \notin A
\end{cases}
\]

and

\[
\nu_A(x) = \begin{cases} 
0.2, & \text{if } x \in A \\
0.7, & \text{for } x \notin A
\end{cases}
\]

Then, we can verify that $A = (\mu_A, \nu_A)$ is both $(\in, \in)$ and $(\in, \in \lor q)$-IF ideal of $R$. Also, we consider $A$, defined as follows:

\[
\mu_A(x) = \begin{cases} 
0.7, & \text{if } x \in A \\
0.2, & \text{for } x \notin A
\end{cases}
\]
and

\[ v_A(x) = \begin{cases} 0.2, & \text{if } x \in A \\ 0.6, & \text{for } x \notin A. \end{cases} \]

Then, it can be easily verified that \( A = (\mu_A, v_A) \) is a \((\in \land q, \notin)\)-IF ideal of \( R \). However, \( A = (\mu_A, v_A) \) is not a \((q, q)\)-IF ideal of \( R \), because if \( x \in A \), \( y \notin A \) and \( s = 0.4 \), \( t = 0.85 \), then \( x + y \notin A \) and \( [x_s qA] \land [y_t qA] = 1 \) but \( [(x_s + y_t) qA] < 1 \). Again, if we take \( \mu_A(x) = 0.4 \) and \( v_A(x) = 0.6 \) for all \( x \in R \), then \( A = (\mu_A, v_A) \) is a \((q, q)\)-IF ideal of \( R \). We note that, in this case \( A \) is not a \((\in, \notin)\)-IF ideal of \( R \).

**Example 3.2.** Consider the ring \( R = \{0, a, b, c\} \) with addition and multiplication operations defined as follows:

\[
\begin{array}{c|ccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & c \\
b & b & c & 0 \\
c & c & b & a \\
\end{array}
\]

and

\[
\begin{array}{c|ccc}
\cdot & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & 0 & 0 & b \\
c & 0 & 0 & b \\
\end{array}
\]

Take \( \mu_A(0) = r \), \( \mu_A(a) = r \), \( \mu_A(b) = s \), \( \mu_A(c) = s \) and \( v_A(0) = 1 - t \), \( v_A(a) = 1 - t \), \( v_A(b) = w \), \( v_A(c) = w \), where \( 0 < s < t < 1 \), \( r \in [0, s) \) and \( w \in [0, 1 - t] \). Then, \( A = (\mu_A, v_A) \) is an intuitionistic fuzzy ideal with thresholds \((s, t)\) of \( R \). However, if we take \( x = b, y = b, \alpha = \in, \beta = \in \) and let \( p, q \in [0, 1] \) be such that \([x_p \alpha A] \land [y_q \alpha A] = 1\), then we have \( s \geq p, s \geq q \). Thus, \( s \geq p \land q \). Since \( x + y = 0 \) so we have \( \mu_A(x + y) = r < s \). Now if \( A \) is a \((\in, \notin)\)-intuitionistic fuzzy ideal of \( R \), then \([x_p + y_q \beta A] \geq [x_p \alpha A] \land [y_q \alpha A] \) implies \( r \geq p \land q \), which will lead to a contradiction if we choose \( r < p, q < s \). Therefore, \( A \) is not a \((\in, \notin)\)-IF ideal of \( R \). Here, we note that \( A \) is not an intuitionistic fuzzy ideal of \( R \) with thresholds \((0, 1)\).

**Definition 3.5.** Let \( A = (\mu_A, v_A) \) be an intuitionistic fuzzy set in \( R \). Then, by the support of \( A \), we mean a crisp subset, \( A^* \) of \( R \), and it is defined as follows:

\[ A^* = \{x \in R \mid \mu_A(x) \lor (1 - v_A(x)) > 0\} \]

That is, \( A^* = \{x \in R \mid A_0(x) > 0\} \).

**Definition 3.6.** Let \( A = (\mu_A, v_A) \) be an intuitionistic fuzzy set in \( R \) and \( \alpha \in [0, 1] \). Then, by a \( \alpha \)-level set of \( A \), we mean a crisp subset, \( A_{\alpha} \) of \( R \), and it is defined as follows:

\[ A_{\alpha} = \{x \in R \mid [x_\alpha \in A] > 0\} \]

**Theorem 3.1.** Let \( A = (\mu_A, v_A) \) be a non-zero \((i.e., \alpha \neq (0, 1))\) \((\alpha, \beta)\)-intuitionistic fuzzy ideal of \( R \). If \( \alpha \neq \in \land q \), then \( A_0 \) is a fuzzy ideal of \( R \).

**Proof.** We show

1. \( A_0(x + y) \geq A_0(x) \land A_0(y) \),
2. \( A_0(-x) \geq A_0(x) \),
(3) \( A_0(xy) \geq A_0(x) \lor A_0(y) \).

Since \((R, +)\) is a group so, (1) and (2) follow from Theorem 4.1 of [23], because \( A \) is also a \((\alpha, \beta)\)-intuitionistic fuzzy subgroup of \((R, +)\).

(I) For (3), first we claim that, \( A_0(x) \lor A_0(y) = 1 \Rightarrow A_0(xy) = 1 \). Let \( A_0(x) \lor A_0(y) = 1 \). Then, \( A_0(x) = 1 \) or \( A_0(y) = 1 \). \( \Rightarrow \mu_A(x) > 0 \) or \( \mu_A(y) > 0 \). Put \( t = \mu_A(x) \lor \mu_A(y) \), then \( t > 0 \). Therefore, we must have \( s \in (0, 1) \) such that \( 0 < 1 - s < t = \mu_A(x) \lor \mu_A(y) \). Now, we have \( t = \mu_A(x) \lor \mu_A(y) \),

\[ \Rightarrow \text{either } \mu_A(x) = t \text{ or } \mu_A(y) = t, \]
\[ \Rightarrow \text{either } A_t(x) = 1 \text{ or } A_t(y) = 1, \]
\[ \Rightarrow \text{either } [x_t \in A] = 1 \text{ or } [y_t \in A] = 1, \]
\[ 1 - s < t = \mu_A(x) \lor \mu_A(y), \]
\[ \Rightarrow \text{either } 1 - s < \mu_A(x) \text{ or } 1 - s < \mu_A(y), \]
\[ \Rightarrow \text{either } A_{1-s}(x) = 1 \text{ or } A_{1-s}(y) = 1, \]
\[ \Rightarrow \text{either } [x_{1-s}qA] = 1 \text{ or } [y_{1-s}qA] = 1. \]

Now,

(i) if \( \alpha \in \varepsilon \), then for \( \beta \in \{\varepsilon, q, \in \land q, \in \lor q\} \) we have from (3) of Definition 3.3

\[ 1 \geq [(x,y)\beta A] \geq [x, \alpha A] \lor [y, \alpha A] = [x_t \in A] \lor [y_t \in A] = 1, \]

because \([x_t \in A] = 1 \text{ or } [y_t \in A] = 1 \). Therefore, \([(x,y)\beta A] = 1 \Rightarrow \text{either } A_t(xy) = 1 \text{ or } A_t(xqA) = 1 \Rightarrow A_t(xy) = 1. \]

(ii) if \( \alpha = \in \lor q \), then for \( \beta \in \{\varepsilon, q, \in \land q, \in \lor q\} \) we have from (3) of Definition 3.3

\[ 1 \geq [(x,y)\beta A] \geq [x, \alpha A] \lor [y, \alpha A] = [x_t \in qA] \lor [y_t \in qA] = [x_t \in qA] \lor [y_t \in qA] \lor [y_t \in A] = 1, \]

because \([x_t \in A] = 1 \text{ or } [y_t \in A] = 1 \). Therefore, \([(x,y)\beta A] = 1, \]
\[ \Rightarrow \text{either } A_t(xy) = 1 \text{ or } A_t(xqA) = 1; \]
\[ \Rightarrow \text{either } \mu_A(xy) > t > 0 \text{ or } \mu_A(xy) > 1 - t > 0; \]
\[ \Rightarrow \mu_A(xy) < 0 \Rightarrow A_0(xy) = 1. \]

(iii) if \( \alpha = q \), then for \( \beta \in \{\varepsilon, q, \in \land q, \in \lor q\} \) we have from (3) of Definition 3.3

\[ 1 \geq [(x,y)\beta A] \geq [x, \alpha A] \lor [y, \alpha A] = [x_t \in qA] \lor [y_t \in qA] \lor [y_t \in qA] = 1, \]

because \([x_t \in A] = 1 \text{ or } [y_t \in A] = 1 \). Therefore, \([(x,y)\beta A] = 1, \]
\[ \Rightarrow \text{either } A_t(xy) = 1 \text{ or } A_t(xqA) = 1 \Rightarrow \text{either } \mu_A(xy) > s > 0 \text{ or } \mu_A(xy) > 1 - s > 0 \Rightarrow \mu_A(xy) > 0 \Rightarrow A_0(xy) = 1. \]

(II) Next we show, \( A_0(x) \lor A_0(y) = 1/2 \Rightarrow A_0(xy) \geq 1/2 \). Let \( A_0(x) \lor A_0(y) = 1/2 \). Then, \( A_0(x) = 1/2 \text{ or } A_0(y) = 1/2 \Rightarrow \overline{v}_A(x) < 1 \text{ or } v_A(y) < 1 \Rightarrow v_A(x) \land v_A(y) < 1 \). So, there exists \( s, t \in (0, 1) \) such that \( v_A(x) \land v_A(y) < 1 - t < s < 1 \). Then

\[ 0 < t < 1 - v_A(x) \land v_A(y) = (1 - v_A(x)) \lor (1 - v_A(y)), \]
\[ \Rightarrow \text{either } \mu_A(x) = 0 < t < 1 - v_A(x) \text{ or } \mu_A(y) = 0 < t < 1 - v_A(y), \]
\[ \Rightarrow \text{either } A_t(x) = 1/2 \text{ or } A_t(y) = 1/2, \]
\[ \Rightarrow \text{either } [x_t \in A] = 1/2 \text{ or } [y_t \in A] = 1/2, \]
\[ \Rightarrow v_A(x) \land v_A(y) < s < 1, \]
\[ \Rightarrow \text{either } v_A(x) < s \leq 1 = 1 - 0 = 1 - \mu_A(x) \text{ or } v_A(y) < s \leq 1 = 1 - 0 = 1 - \mu_A(y), \]
Theorem 3.4. An IFS $A = (\mu_A, \nu_A)$ of $R$ is a $(\in, \in)$-intuitionistic fuzzy ideal of $R$ if and only if $A$ is an intuitionistic fuzzy ideal of $R$ with thresholds $(0, 1)$. 

Proof. Suppose that $A = (\mu_A, \nu_A)$ is a $(\in, \in)$-intuitionistic fuzzy ideal of $R$. To show $A$ is an intuitionistic fuzzy ideal of $R$ with thresholds $(0, 1)$ i.e. to show
(1) \( \mu_A(x+y) \geq \mu_A(x) \land \mu_A(y) \);
(2) \( \mu_A(-x) \geq \mu_A(x) \);
(3) \( \mu_A(xy) \geq \mu_A(x) \lor \mu_A(y) \);
(4) \( v_A(x+y) \leq v_A(x) \lor v_A(y) \);
(5) \( v_A(-x) \leq v_A(x) \);
(6) \( v_A(xy) \leq v_A(x) \land v_A(y) \), for all \( x, y \in R \).

For (1), let \( t = \mu_A(x) \land \mu_A(y) \). Then, \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \), which implies that \( A_t(x) = 1 \) and \( A_t(y) = 1 \), and so \( [x_t \in A] = 1 \) and \( [y_t \in A] = 1 \). Now \( 1 \geq [x_t + y_t \in A] \geq [x_t \in A] \land [y_t \in A] = 1 \Rightarrow [(x_t + y_t) \in A] = 1 \Rightarrow \mu_A(x+y) \geq t = \mu_A(x) \land \mu_A(y) \).

In a similar manner we can prove (2).

(3) Let \( t = \mu_A(x) \lor \mu_A(y) \), then either \( \mu_A(x) = t \) or \( \mu_A(y) = t \), which implies either \( A_t(x) = 1 \) or \( A_t(y) = 1 \), and so either \( [x_t \in A] = 1 \) or \( [y_t \in A] = 1 \). Now \( 1 \geq [x_t, y_t \in A] \geq [x_t \in A] \lor [y_t \in A] = 1 \Rightarrow [(x_t, y_t) \in A] = 1 \Rightarrow \mu_A(xy) \geq t = \mu_A(x) \lor \mu_A(y) \).

(4) If \( v_A(x+y) = 0 \), then it is obvious. Let \( s = v_A(x+y) > 0 \) and let \( t \in [0,1] \) be such that \( t > 1 - s = 1 - v_A(x+y) \), then we have \( 0 = [(x_t+y_t) \in A] \geq [x_t \in A] \land [y_t \in A] \Rightarrow [x_t \in A] \land [y_t \in A] = 0 \Rightarrow [x_t \in A] = 0 \) or \( [y_t \in A] = 0 \) i.e., either \( t > 1 - v_A(x) \) or \( t > 1 - v_A(y) \) \( \Rightarrow \) either \( v_A(x) > 1-t \) or \( v_A(y) > 1-t \) \( \Rightarrow v_A(x) \lor v_A(y) > 1-t \). Therefore, \( v_A(x) \lor v_A(y) \geq \forall \{1-t \mid t > 1-s \} = \forall \{1-t \mid s > 1-t \} = s = v_A(x+y) \). Thus, \( v_A(x+y) \leq v_A(x) \lor v_A(y) \).

Similarly, we have (5).

Lastly, if \( v_A(xy) = 0 \), then it is obvious. Let \( s = v_A(xy) > 0 \) and let \( t \in [0,1] \) be such that \( t > 1-s = 1 - v_A(xy) \), then we have \( 0 = [(x_t y_t) \in A] \geq [x_t \in A] \lor [y_t \in A] \Rightarrow [x_t \in A] \lor [y_t \in A] = 0 \Rightarrow [x_t \in A] = 0 \) and \( [y_t \in A] = 0 \) i.e., \( t > 1 - v_A(x) \) and \( t > 1 - v_A(y) \) \( \Rightarrow v_A(x) > 1-t \) and \( v_A(y) > 1-t \) \( \Rightarrow v_A(x) \land v_A(y) > 1-t \). Therefore, \( v_A(x) \land v_A(y) \geq \forall \{1-t \mid t > 1-s \} = \forall \{1-t \mid s > 1-t \} = s = v_A(xy) \). Thus, \( v_A(xy) \leq v_A(x) \land v_A(y) \).

Conversely, we assume \( A \) is an intuitionistic fuzzy ideal of \( R \) with thresholds \( \{0,1\} \).
We need to show \( A = (\mu_A, v_A) \) is a \((\in,\notin)\)-intuitionistic fuzzy ideal of \( R \). Let \( x, y \in R \) and \( s, t \in [0,1] \).
Let \( a = [x_t \in A] \land [y_t \in A] \).

Case I. \( a = 1 \). Then, \([x_t \in A] = 1 \) and \([y_t \in A] = 1 \Rightarrow \mu_A(x) \geq s \) and \( \mu_A(y) \geq t \Rightarrow \mu_A(x+y) \geq s \land t \Rightarrow [(x_t+y_t) \in A] = 1 \geq 1 \in [x_t \in A] \land [y_t \in A] \).

Case II. \( a = 1/2 \). Then, \([x_t \in A] \geq 1/2 \) and \([y_t \in A] \geq 1/2 \Rightarrow 1 - v_A(x) \geq s \) and \( 1 - v_A(y) \geq t \Rightarrow 1 - v_A(x+y) \geq 1 - v_A(x) \land v_A(y) = (1 - v_A(x)) \land (1 - v_A(y)) \geq s \land t \Rightarrow [(x_t+y_t) \in A] \geq 1/2 \in [x_t \in A] \land [y_t \in A] \).

Case III. \( a = 0 \). Then, the result is obvious. Thus, in all cases we have \( [(x_t+y_t) \in A] \geq [x_t \in A] \land [y_t \in A] \). In a similar manner we can prove that \( [-x_t \in A] \geq [x_t \in A] \).
Let \( b = [x_t \in A] \lor [y_t \in A] \).

Case I. \( b = 1 \). Then, either \([x_t \in A] = 1 \) or \([y_t \in A] = 1 \Rightarrow \) either \( \mu_A(x) \geq s \) or \( \mu_A(y) \geq t \Rightarrow \mu_A(xy) \geq s \lor t \Rightarrow [(x_t y_t) \in A] = 1 \geq 1 \in [x_t \in A] \lor [y_t \in A] \).

Case II. \( b = 1/2 \). Then, either \([x_t \in A] \leq 1/2 \) or \([y_t \in A] \leq 1/2 \) \( \Rightarrow \) either \( 1 - v_A(x) \geq s \) or \( 1 - v_A(y) \geq t \Rightarrow 1 - v_A(x+y) \geq 1 - v_A(x) \land v_A(y) = (1 - v_A(x)) \lor (1 - v_A(y)) \geq s \lor t \Rightarrow [(x_t y_t) \in A] \geq 1/2 \in [x_t \in A] \lor [y_t \in A] \). Hence, \( A \) is a \((\in,\notin)\)-intuitionistic fuzzy ideal of \( R \).

As a consequence of Theorem 3.3 and Theorem 3.4, we have the following:

**Theorem 3.5.** If an IFS \( A = (\mu_A, v_A) \) of \( R \) is a \((\in,\notin)\)-intuitionistic fuzzy ideal of \( R \), then for any \( p \in (0,1] \), \( A_p \) is an ideal of \( R \).
Theorem 3.6. An IFS $A = (\mu_A, \nu_A)$ of $R$ is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of $R$ if and only if $A$ is an intuitionistic fuzzy ideal of $R$ with thresholds $(0, 0.5)$.

Proof. Suppose that $A = (\mu_A, \nu_A)$ is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of $R$. To show $A$ is an intuitionistic fuzzy ideal of $R$ with thresholds $(0, 0.5)$ i.e. to show

\begin{enumerate}
  \item $\mu_A(x + y) \geq (\mu_A(x) \land \mu_A(y)) \land 0.5$;
  \item $\mu_A(-x) \geq \mu_A(x) \land 0.5$;
  \item $\mu_A(xy) \geq (\mu_A(x) \lor \mu_A(y)) \lor 0.5$;
  \item $\nu_A(x + y) \leq (\nu_A(x) \lor \nu_A(y)) \lor 0.5$;
  \item $\nu_A(-x) \leq \nu_A(x) \lor 0.5$;
  \item $\nu_A(xy) \leq (\nu_A(x) \lor \nu_A(y)) \lor 0.5$, for all $x, y \in R$.
\end{enumerate}

For (1), let $t = (\mu_A(x) \land \mu_A(y)) \land 0.5$, then $\mu_A(x) \geq t, \mu_A(y) \geq t \Rightarrow [x_t \in A] = 1, [y_t \in A] = 1$. Therefore, from (1) of Definition 3.4 we have $1 \geq [(x_t + y_t) \in qA] \geq [x_t \in A] \land [y_t \in A] = 1.$

Thus, $[(x_t + y_t) \in qA] = 1,$

\begin{align*}
\Rightarrow & [(x_t + y_t) \in A] \lor [(x_t + y_t)qA] = 1, \\
\Rightarrow & [(x_t + y_t) \in A] = 1 \text{ or } [(x_t + y_t)qA] = 1, \\
\Rightarrow & \mu_A(x + y) \geq t \text{ or } \mu_A(x + y) + t > 1, \\
\Rightarrow & \mu_A(x + y) \geq t \text{ or } \mu_A(x + y) > 1 - t \geq 0.5 \geq t, \\
\Rightarrow & \mu_A(x + y) \geq t \text{ or } \mu_A(x + y) = t \land \land 0.5.
\end{align*}

Similarly, we can prove (2).

(3) Let $t = (\mu_A(x) \lor \mu_A(y)) \lor 0.5 = (\mu_A(x) \lor 0.5) \lor (\mu_A(y) \lor 0.5)$. This implies that $(\mu_A(x) \lor 0.5) = t \text{ or } (\mu_A(y) \lor 0.5) = t \Rightarrow \mu_A(x) \lor t \text{ or } \mu_A(y) \lor t \Rightarrow [x_t \in A] = 1 \text{ or } [y_t \in A] = 1.$ Therefore, from (3) of Definition 3.4 we have $1 \geq [(x_t y_t) \in \in \vee qA] \geq [x_t \in A] \lor [y_t \in A] = 1.$ Thus, $[(x_t y_t) \in \in \vee qA] = 1,$

\begin{align*}
\Rightarrow & [(x_t y_t) \in A] = 1 \text{ or } [(x_t y_t)qA] = 1, \\
\Rightarrow & \mu_A(xy) \geq t \text{ or } \mu_A(xy) + t > 1, \\
\Rightarrow & \mu_A(xy) \geq t \text{ or } \mu_A(xy) > 1 - t \geq 0.5 \geq t, \\
\Rightarrow & \mu_A(xy) \geq t \text{ or } \mu_A(xy) = t \land \land 0.5.
\end{align*}

(4) Let $\nu_A(x) \lor \nu_A(y) \lor 0.5 = 1 - s.$ Then, $\nu_A(x) \leq 1 - s$ and $\nu_A(y) \leq 1 - s \Rightarrow s \leq 1 - \nu_A(x)$ and $s \leq 1 - \nu_A(y) \Rightarrow [x_s \in A] \geq 1/2$ and $[y_s \in A] \geq 1/2$. Therefore, from (1) of definition 3.4 we have, $1 \geq [(x_t + y_t) \in qA] \geq [x_t \in A] \land [y_t \in A] \geq 1/2$. This implies that $[(x_t + y_t) \in A] \lor [(x_t + y_t)qA] \geq 1/2,$

\begin{align*}
\Rightarrow & [(x_t + y_t) \in A] \geq 1/2 \text{ or } [(x_t + y_t)qA] \geq 1/2, \\
\Rightarrow & \text{either } s \leq 1 - \nu_A(x) \lor \nu_A(x + y) < s \leq 1 - s, \text{[since } 1 - s \geq 0.5 \text{ so, } s \leq 0.5] \\
\Rightarrow & \nu_A(x + y) \leq 1 - s = \nu_A(x) \lor \nu_A(y) \lor 0.5.
\end{align*}

Similarly, we can prove (5).

(6) Let $(\nu_A(x) \lor \nu_A(y)) \lor 0.5 = 1 - s$. Then

\begin{align*}
1 - (\nu_A(x) \lor 0.5) \lor (\nu_A(y) \lor 0.5) = s, \\
\Rightarrow & (1 - \nu_A(x) \lor 0.5) \lor (1 - \nu_A(y) \lor 0.5) = s, \\
\Rightarrow & ((1 - \nu_A(x)) \lor 0.5) \lor ((1 - \nu_A(y)) \lor 0.5) = s, \\
\Rightarrow & (1 - \nu_A(x)) \lor 0.5 = s \text{ or } (1 - \nu_A(y)) \lor 0.5 = s, \\
\Rightarrow & (1 - \nu_A(x)) \geq s \text{ or } (1 - \nu_A(y)) \geq s, \\
\Rightarrow & [x_s \in A] \geq 1/2 \text{ or } [y_s \in A] \geq 1/2, \\
\Rightarrow & [x_s y_s \in qA] \geq [x_s \in A] \lor [y_s \in A] \geq 1/2, \text{[By (3) of Definition 3.4]} \\
\Rightarrow & [x_s y_s \in qA] \geq 1/2, \\
\Rightarrow & [x_s y_s \in A] \geq 1/2 \text{ or } [x_s y_s qA] \geq 1/2,
\end{align*}
Therefore, we must have \((x_s + y_t) ∈ \mathbb{Q}A\) = 1.

Case II. \(a = 1/2\). Then, \([x_s ∈ A] ≥ 1/2\) and \([y_t ∈ A] ≥ 1/2\) which implies that \(1 − v_A(x) ≥ s\) and \(1 − v_A(y) ≥ t\).

\[
1 − v_A(x) ∨ v_A(y) = (1 − v_A(x)) ∧ (1 − v_A(y)) ≥ s ∧ t
\]

If \([(x_s + y_t) ∈ \mathbb{Q}A = 0\), then \((1 − v_A(x + y)) < s ∧ t\) and \(v_A(x + y) ≥ s ∧ t\). Now, from \(0 < v_A(x + y) ≤ v_A(x) ∨ v_A(y) \lor 0.5\), we get \(v_A(x + y) ≤ v_A(x) ∨ v_A(y)\) and \(1 − v_A(x + y) ≥ 1 − v_A(x) ∨ v_A(y) = (1 − v_A(x)) ∧ (1 − v_A(y)) ≥ s ∧ t\), which contradicts \((1 − v_A(x + y)) < s ∧ t\). Therefore, we must have \([(x_s + y_t) ∈ \mathbb{Q}A] ≥ 1/2 = [x_s ∈ A] ∧ [y_t ∈ A]\).

Case III. \(a = 0\). Then, the result is obvious. Thus, in all cases, \([(x_s + y_t) ∈ \mathbb{Q}A] ≥ [x_s ∈ A] ∧ [y_t ∈ A]\).

Similarly, we can prove that \([-x_s ∈ \mathbb{Q}A] ≥ [x_s ∈ A]\).

Next, we claim that \([(x_s y_t) ∈ \mathbb{Q}A] ≥ [x_s ∈ A] ∨ [y_t ∈ A]\). Let \(b = [x_s ∈ A] ∨ [y_t ∈ A]\).

Case I. \(b = 1\). Then, either \([x_s ∈ A] = 1\) or \([y_t ∈ A] = 1\), which implies either \(μ_A(x) ≥ s\) or \(μ_A(y) ≥ t\). If \([x_s y_t ∈ \mathbb{Q}A] ≤ 1/2\), then \([x_s y_t ∈ A] ≤ 1/2\) and \([x_s y_t ∈ a] ≤ 1/2 \Rightarrow μ_A(xy) < s ∨ t\) and \(s ∨ t ≤ 1 − μ_A(xy) \Rightarrow μ_A(xy) < s ∨ t\) and \(μ_A(xy) ≤ 1 − s ∨ t\). Now, \(0.5 ≥ μ_A(xy) ≥ (μ_A(x) ∨ μ_A(y)) \lor 0.5\) implies \(μ_A(xy) ≥ μ_A(x) ∨ μ_A(y) ≥ s ∨ t\), which contradicts \(μ_A(xy) < s ∨ t\). Therefore, we must have \([x_s y_t ∈ \mathbb{Q}A] = 1\).

Case II. \(b = 1/2\). Then, either \([x_s ∈ A] = 1/2\) or \([y_t ∈ A] = 1/2\), which implies either \(s ≤ 1 − v_A(x)\) or \(t ≤ 1 − v_A(y)\). If \([x_s y_t ∈ \mathbb{Q}A] = 0\), then \([x_s y_t ∈ A] = 0\) and \([x_s y_t ∈ a] = 0 \Rightarrow s ∨ t ≥ 1 − v_A(xy)\) and \(s ∨ t ≤ v_A(xy) \Rightarrow v_A(xy) < 1 − s ∨ t\) and \(v_A(xy) ≤ v_A(x) ∨ v_A(y) = 0.5 < v_A(xy) = (v_A(x) ∧ v_A(y)) ∨ 0.5 = v_A(x) ≤ v_A(x) ∧ v_A(y)\). Now, \(1 − v_A(xy) ≥ 1 − v_A(x) ∨ v_A(y) = (1 − v_A(x)) ∨ (1 − v_A(y)) ≥ s ∨ t\), a contradiction to \(s ∨ t > 1 − v_A(xy)\). Therefore, we have \([x_s y_t ∈ \mathbb{Q}A] ≥ 1/2 = [x_s ∈ A] ∨ [y_t ∈ A]\). Hence, \([x_s y_t ∈ \mathbb{Q}A] ≥ [x_s ∈ A] ∨ [y_t ∈ A]\).

As a consequence of Theorem 3.3 and Theorem 3.6, we have the following:

**Theorem 3.7.** If an IFS \(A = (μ_A, v_A)\) of \(R\) is a \((∈, ∈ ∨ q)\)-intuitionistic fuzzy ideal of \(R\), then for any \(p ∈ (0.5, 1)\), \(A_p\) is an ideal of \(R\).

**Theorem 3.8.** An IFS \(A = (μ_A, v_A)\) of \(R\) is a \((∈ ∨ q, ∈)\)-intuitionistic fuzzy ideal of \(R\) if and only if \(A\) is an intuitionistic fuzzy ideal of \(R\) with thresholds \((0.5, 1)\).

**Proof.** Suppose that \(A = (μ_A, v_A)\) is a \((∈ ∨ q, ∈)\)-intuitionistic fuzzy ideal of \(R\). To show

1. \(μ_A(x + y) ∨ 0.5 ≥ μ_A(x) ∧ μ_A(y);\)
2. \(μ_A(−x) ∨ 0.5 ≥ μ_A(x);\)
3. \(μ_A(xy) ∨ 0.5 ≥ μ_A(x) ∨ μ_A(y);\)
4. \(v_A(x + y) ∧ 0.5 ≤ v_A(x) ∨ v_A(y);\)
5. \(v_A(−x) ∧ 0.5 ≤ v_A(x);\)
(6) $v_A(xy) \wedge 0.5 \leq v_A(x) \wedge v_A(y)$, for all $x, y \in R$.

Let $x, y \in R$ and $t = μ_A(x) \wedge μ_A(y)$. If $μ_A(x + y) \lor 0.5 < t = μ_A(x) \wedge μ_A(y)$, then

$μ_A(x) > 0.5$ and $μ_A(y) \geq t > 0.5$,

$⇒ [x_t ∈ A] = 1, [x_t q A] = 1, [y_t ∈ A] = 1, [y_t q A] = 1$,

$⇒ [x_t ∈ q A] = 1, [y_t ∈ q A] = 1$,

$⇒ [x_t ∈ q A] \wedge [y_t ∈ q A] = 1$.

Therefore, $[(x_t + y_t) ∈ A] \geq [x_t ∈ q A] \wedge [y_t ∈ q A] = 1$, which gives $[(x_t + y_t) ∈ A] = 1 ⇒ μ_A(x + y) \geq t$, a contradiction to our assumption $μ_A(x + y) \leq μ_A(x + y) \land 0.5 < t$. Therefore, we have $μ_A(x + y) \lor 0.5 \geq t = μ_A(x) \wedge μ_A(y)$.

Similarly, we can prove that $μ_A(−x) \lor 0.5 \geq μ_A(x)$.

Next, let $t = μ_A(x) \lor μ_A(y)$, then $μ_A(x) = t$ or $μ_A(y) = t$. If $μ_A(xy) \lor 0.5 < t$, then either $μ_A(x) = t > 0.5$ or $μ_A(y) = t > 0.5$, which implies that $[x_t ∈ q A] = 1$, or $[y_t ∈ q A] = 1$.

Now

$[(x_t y_t) ∈ A] \geq [x_t ∈ q A] \lor [y_t ∈ q A] = 1$.

From which we get $[(x_t y_t) ∈ A] = 1 ⇒ μ_A(xy) \geq t$, which contradicts to our assumption $μ_A(xy) < t$. Therefore, we must have $μ_A(xy) \lor 0.5 \geq t = μ_A(x) \lor μ_A(y)$.

(4) Let $t = 1 − s = v_A(x) \lor v_A(y)$, then $1 − s ≥ v_A(x), 1 − s ≥ v_A(y)$. If $v_A(x + y) \lor 0.5 > t$, then we have $s \leq 1 − v_A(x), s \geq 1 − v_A(y), v_A(x + y) > t$ and $s > 0.5 > t$, and so $[x_t ∈ A] \geq 1/2, [x_t q A] \geq 1/2, v_A(x + y) > t$ and $s > 0.5 > t$. Also, $v_A(x) ≤ t < s$ and $v_A(y) ≤ t < s$ imply $[x_t q A] \geq 1/2, [y_t q A] \geq 1/2$. Therefore, from $[(x_t + y_t) ∈ A] \geq [x_t ∈ q A] \wedge [y_t ∈ q A] \lor q A] \geq 1/2$ we have $[(x_t + y_t) ∈ A] ≥ 1/2$. This implies that $s ≤ 1 − v_A(x + y)$, which is a contradiction to $v_A(x + y) > t = 1 − s$. Hence, $v_A(x + y) \lor 0.5 \leq t = v_A(x) \lor v_A(y)$.

Similarly, we can prove that $v_A(−x) \lor 0.5 \leq v_A(x)$.

(6) Let $t = 1 − s = v_A(x) \land v_A(y)$. Then

$s = (1 − v_A(x)) \lor (1 − v_A(y))$,

$⇒ s = 1 − v_A(x) \lor s = 1 − v_A(y)$,

$⇒ [x_t ∈ A] ≥ 1/2 \lor [y_t ∈ A] ≥ 1/2$.

If $v_A(xy) \land 0.5 > t$, then $v_A(xy) > t$ and $t < 0.5 < s$. Therefore, $s = 1 − v_A(x)$ or $s = 1 − v_A(y)$ which implies that $v_A(x) = 1 − s = t < s$ or $v_A(y) = 1 − s = t < s ⇒ [x_t q A] \geq 1/2$ or $[y_t q A] ≥ 1/2$. Thus, we have

$[x_t ∈ A] ≥ 1/2 \lor [y_t ∈ A] ≥ 1/2 \lor [x_t q A] \geq 1/2$ or $[y_t q A] ≥ 1/2$.

Now if $[x_t ∈ A] ≥ 1/2$ and $[x_t q A] = 0$, then we get $s ≤ 1 − v_A(x)$ and $s ≤ v_A(x) ⇒ v_A(x) ≤ 1 − s = t < s$, (since $t < 0.5 < s$), which contradicts to $v_A(x) ≥ s$.

Therefore, $[x_t ∈ A] ≥ 1/2$ and $[x_t q A] = 0$ can’t hold simultaneously. Thus, if $[x_t ∈ A] ≥ 1/2$, then $[x_t q A] ≥ 1/2$.

Similarly, if $[y_t ∈ A] ≥ 1/2$, then $[y_t q A] ≥ 1/2$.

Again, if $[x_t q A] ≥ 1/2$ and $[x_t ∈ A] = 0$, then we get $v_A(x) < s$, $s > 1 − v_A(x)$. Therefore, $s > v_A(x) > 1 − s$, which is true for all $s > 0.5 > t$. Hence, we must have, $v_A(x) = 0.5$.

Similarly, if $[y_t q A] ≥ 1/2$ and $[y_t ∈ A] = 0$, then $v_A(y) = 0.5$. Now, $t = v_A(x) \land v_A(y) = 0.5$, which contradicts to $t < 0.5$. Therefore, we must have

$[x_t q A] ≥ 1/2$ and $[x_t ∈ A] ≥ 1/2$ or $[y_t q A] ≥ 1/2$ and $[y_t ∈ A] ≥ 1/2$.

Thus, if $[x_t ∈ A] ≥ 1/2$, then $[x_t q A] ≥ 1/2$ and vice versa.

or, if $[y_t ∈ A] ≥ 1/2$, then $[y_t q A] ≥ 1/2$ and vice versa.

Thus, in all cases, we have

$[(x_t y_t) ∈ A] ≥ [x_t ∈ q A] \lor [y_t ∈ q A]$,

$⇒ [(x_t y_t) ∈ A] ≥ (([x_t ∈ A] \lor [y_t ∈ A]) \land ([x_t q A] \lor [y_t q A])) \land ([x_t ∈ A] \lor [y_t q A])$.
Theorem 3.10. An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$, is an intuitionistic fuzzy ideal with thresholds $(0.5, 1)$ if and only if for any $a \in [0, 1]$, $A_a$ is a fuzzy ideal of $R$.

Proof. Suppose that $A$ is a $(\epsilon, \epsilon)$-intuitionistic fuzzy ideal of $R$. Let $x, y \in R$ and $a \in [0, 1]$. Then

$$A_a(x + y) = [(x + y)_a] R = [(x + y)_a] \in A = [(x_a + y_a)] A \geq [x_a] A \wedge [y_a] A = A_a(x) \wedge A_a(y),$$

$$A_a(-x) = [-x_a] A \geq [x_a] A = A_a(x),$$

$$A_a(xy) = [(xy)_a] A = [(x_a y_a)_a] A \geq [x_a] A \vee [y_a] A = A_a(x) \vee A_a(y).$$

Hence, $A_a$ is a fuzzy ideal of $R$.

Conversely, we assume $A$ is an intuitionistic fuzzy ideal with thresholds $(0.5, 1)$. Let $x, y \in R$, $s, t \in [0, 1]$ and $a = [x_t \in \land qA] \wedge [y_t \in qA]$. Then

Case I. $a = 1$. Then, $\mu_A(x) / s, \mu_A(x) + s / 1, \mu_A(y) / t, \mu_A(y) + t / 1 > 1$. This implies that $\mu_A(x) / s \geq 0.5$ and $\mu_A(y) / t \geq 0.5$. Now, we have $\mu_A(x + y) / s \geq \mu_A(x) \wedge \mu_A(y) \geq s \land t$, from which we get $[(x + y)_a] A = 1$.

Case II. $a = 1/2$. Then, $s \leq 1 - \nu_A(x), \nu_A(x) < s, t \leq 1 - \nu_A(y), \nu_A(y) < t,$

$$\Rightarrow 1 - \nu_A(x) \geq s > \nu_A(x), 1 - \nu_A(y) \geq t > \nu_A(y),$$

$$\Rightarrow \nu_A(x) < 0.5, \nu_A(x) < 0.5.$$

Therefore, $\nu_A(x + y) / s \geq 0.5 \leq 1 - \nu_A(x) \vee \nu_A(y) \Rightarrow 1 - \nu_A(x + y) / s \geq 0.5 \leq \nu_A(x) \vee \nu_A(y)$ which implies that $1 - \nu_A(x + y) / s \geq 1 - \nu_A(x) / s \wedge (1 - \nu_A(y)) / s \geq s \land t$. Hence, $[(x + y)_a] A / 1/2$. Hence, $[(x + y)_a] A \geq [x_s \in \land qA] \wedge [y_t \in qA]$. We will prove

Case II. $b = 1$. Then, either $\mu_A(x) \geq s, \mu_A(x) + s > 1$ or $\mu_A(y) \geq t, \mu_A(y) + t > 1$. This implies, either $\mu_A(x) / s \geq 0.5$ or $\mu_A(y) / t \geq 0.5$. Now,

$$\mu_A(x) / s \geq 0.5 \lor \mu_A(y) / t \geq 0.5, \text{ from which we get } [(x + y)_a] A = 1.$$
Thus, if \( A_t(y) \geq 1/2 \), then \( t \leq 1 - v_A(xy) \). Therefore, \( [x,y_t \in A] \geq 1/2 \). Thus, in all cases, we get \([x,y_t \in A] \geq [x \in A] \vee [y_t \in A] \).

\[ A_s(xy) \geq 1/2, \text{ and so } s \leq 1 - v_A(xy). \text{ Similarly, if } A_t(y) \geq 1/2, \text{ then } t \leq 1 - v_A(xy). \text{ Thus, } s \vee t \leq 1 - v_A(xy), \text{ which implies that } A_{s \vee t}(xy) \geq 1/2. \text{ Hence, } [x,y_t \in A] \geq 1/2. \text{ Thus, in all cases, we get } [x,y_t \in A] \geq [x \in A] \vee [y_t \in A]. \]

**Theorem 3.11.** An intuitionistic fuzzy set, \( A = (\mu_A, v_A) \) of \( R \) is a \((\epsilon, \epsilon \vee q)\)-intuitionistic fuzzy ideal of \( R \) if and only if for any \( a \in [0,0.5], A_a \) is a fuzzy ideal of \( R \).

**Proof.** Suppose that \( A \) is a \((\epsilon, \epsilon \vee q)\)-intuitionistic fuzzy ideal of \( R \). Then, for any \( a \in (0,0.5) \) and \( x,y \in R \), we have

\[ [x_ay_a \in \vee q] \geq [x_a \in A] \vee [y_a \in A] \Rightarrow A_a(xy) \vee A_{[\frac{a}{2}]}(xy) \geq A_a(x) \vee A_a(y). \]

Since \( 0 < a \leq 0.5 \), therefore we have \( a \leq 0.5 \leq 1 - a \). Then

\[ A_{[\frac{a}{2}]}(xy) = A_{1-a}(xy) \leq A_a(xy) \leq A_a(x) \vee A_a(y). \]

Therefore, \( A_a(x) \vee A_a(y) \leq A_a(xy) \vee A_{[\frac{a}{2}]}(xy) \leq A_a(xy) \vee A_a(xy) = A_a(xy) \), and so \( A_a(xy) \geq A_a(x) \vee A_a(y) \). Similarly, we can prove that \( A_a(x \vee y) \geq A_a(x) \vee A_a(y) \) and \( A_a(-x) \geq A_a(x) \). Therefore, for any \( a \in [0,0.5], A_a \) is a fuzzy ideal of \( R \).

Conversely, we assume for any \( a \in [0,0.5], A_a \) is a fuzzy ideal of \( R \). Let \( s,t \in [0,1] \) and \( x,y \in R \).

1. If \( s \wedge t \leq 0.5 \), then let \( a = [x_s \in A] \wedge [y_t \in A] \).

**Case I.** \( a = 1 \). Then, \( A_s(x) = 1 \) and \( A_t(y) = 1 \), and so \( A_{s \vee t}(x+y) \geq A_{s \vee t}(x) \wedge A_{s \vee t}(y) \geq A_s(x) \wedge A_t(y) = 1 \). Therefore, we have \( A_{s \vee t}(x+y) = 1 \Rightarrow [(x_s + y_t) \in A] = 1 \). Now, \([x_s + y_t] \in \vee qA] = [(x_s + y_t) \in A] \vee [(x_s + y_t)qA] = 1 \).

**Case II.** \( a = 1/2 \). Then, \( A_s(x) \geq 1/2 \) and \( A_t(y) \geq 1/2 \), and so \( A_{s \vee t}(x+y) \geq A_{s \vee t}(x) \wedge A_{s \vee t}(y) \geq A_s(x) \wedge A_t(y) = 1 \). Therefore, we have \( A_{s \vee t}(x+y) = 1 \Rightarrow [(x_s + y_t) \in A] \geq 1/2 \). Now, \([(x_s + y_t) \in \vee qA] = [(x_s + y_t) \in A] \vee [(x_s + y_t)qA] \geq 1/2 \). Therefore, \([x_s + y_t] \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A] \).

If \( s \wedge t > 0.5 \), then let \( a \in (0,1) \) such that \( 1 - s \wedge t < a < 0.5 < s \wedge t \). Now, \( A_{[\frac{a}{2}]}(x+y) = A_{1-a}(x+y) \geq A_{s \vee t}(x+y) \) and \( A_{[\frac{a}{2}]}(x+y) = A_{1-s}(x+y) \geq A_a(x+y) \).

Therefore, \([(x_s + y_t) \in \vee qA] = [(x_s + y_t) \in A] \vee [(x_s + y_t)qA] = A_{s \vee t}(x+y) \vee A_{[\frac{a}{2}]}(x+y) = A_{[\frac{a}{2}]}(x+y) \geq A_a(x+y) \geq A_a(x) \wedge A_a(y) \geq A_a(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A] \), and hence \([x_s + y_t] \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A] \).

Similarly, we can prove that \([-x_s \in \vee qA] \geq [x_s \in A] \).

3. If \( s \vee t \leq 0.5 \), then let \( b = [x_s \in A] \vee [y_t \in A] \).

**Case I.** \( b = 1 \). Then, \( A_s(x) = 1 \) or \( A_t(y) = 1 \). If \( A_s(x) = 1 \), then \( A_s(xy) \geq A_s(x) \vee A_s(y) = 1 \), and so \( A_s(xy) = 1 \). This implies that \( \mu_A(xy) \geq s \). Similarly, if \( A_t(y) = 1 \), then \( \mu_A(xy) \geq t \). Therefore, we obtain \( \mu_A(xy) \geq s \vee t \), from which we get \([x_s y_t] \in A] = 1 \).

Thus, \([x_s y_t] \in \vee qA] = [(x_s y_t) \in A] \vee [(x_s y_t)qA] = 1 \).

**Case II.** \( b = 1/2 \). Then, \( A_s(x) = 1/2 \) or \( A_t(y) = 1/2 \). If \( A_s(x) = 1/2 \), then \( A_s(xy) \geq A_s(x) \vee A_s(y) = 1/2 \), and so \( s \leq 1 - v_A(xy) \). Similarly, if \( A_t(y) = 1/2 \), then \( t \leq 1 - v_A(xy) \). Therefore, we have \( s \vee t \leq 1 - v_A(xy) \) which implies that \( A_{s \vee t}(xy) \geq 1/2 \).

Thus, \([x_s y_t] \in A] \geq 1/2 \), and so \([x_s y_t] \in \vee qA] = [(x_s y_t) \in A] \vee [(x_s y_t)qA] = 1/2 \). Therefore, \([x_s y_t] \in \vee qA] \geq [x_s \in A] \vee [y_t \in A] \).

If \( s \vee t > 0.5 \), then let \( a \in (0,1) \) be such that \( 1 - s \vee t < a < 0.5 < s \vee t \). Now, \( A_{[\frac{a}{2}]}(xy) = A_{1-s}(xy) \geq A_{s \vee t}(xy) \), and \( A_{[\frac{a}{2}]}(xy) = A_{1-s}(xy) \geq A_a(xy) \).

Therefore, \([x_s y_t] \in \vee qA] = [(x_s y_t) \in A] \vee [(x_s y_t)qA] = A_{s \vee t}(xy) \vee A_{[\frac{a}{2}]}(xy) = A_{[\frac{a}{2}]}(xy) \geq A_a(xy) \geq A_a(x) \vee A_a(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A], \) and hence \([x_s y_t] \in \vee qA] \geq [x_s \in A] \vee [y_t \in A]. \)
Theorem 3.12. An intuitionistic fuzzy set, \( A = (\mu_A, \nu_A) \) of \( R \) is a \((\in, \in, \not\in)\)-intuitionistic fuzzy ideal of \( R \) if and only if for any \( a \in (0.5, 1] \), \( A_a \) is a fuzzy ideal of \( R \).

Proof. Suppose that \( A \) is a \((\in, \in, \not\in)\)-intuitionistic fuzzy ideal of \( R \). Let \( a \in (0.5, 1] \) and \( x, y \in R \), then \( A_{a|x}(x) \geq A_{a|y}(y) \). Thus, 
\[
A_a(x + y) = [x_a + y_a] = [x_a \in \land qA] \land [y_a \in \land qA]
\]

\[
= A_a(x) \land A_{a|x}(x) \land A_a(y) \land A_{a|y}(y) = A_a(x) \land A_a(y).
\]

Therefore, \( A_a(x + y) \geq A_a(x) \land A_a(y) \). Similarly, we have \( A_a(-x) \geq A_a(x) \).

Therefore, \( A_a(xy) \geq A_a(x) \lor A_a(y) \).

Conversely, we assume for any \( a \in (0.5, 1] \), \( A_a \) is a fuzzy ideal of \( R \). Let \( x, y \in R, s, t \in (0, 1] \).

(1) Let \( b = \{x \in \land qA \mid y \in \land qA\} \).

Case I. \( b = 1 \). Then, \( \mu_a(x) \geq s, \mu_a(y) > -1 \), \( \mu_a(y) \geq t \). Therefore, \( \mu_a(x) > 0.5, \mu_a(y) > 0.5 \). Let \( a = \mu_a(x) \land \mu_a(y) \). Then, \( a > 0.5 \) and \( \mu_a(x) \geq a, \mu_a(y) \geq a \), and so \( A_a(x) = 1, A_a(y) = 1 \). Thus, \( A_a(x + y) \geq A_a(x) \land A_a(y) = 1 \) implies \( A_a(x + y) = 1 \), and so \( \mu_a(x + y) \geq a = \mu_a(x) \land \mu_a(y) \geq s \land t \). Therefore, \( \{x + y \mid A \} = 1 \).

Case II. \( b = 1/2 \). Then, \( 1 - v_a(x) \geq s, s > v_a(x) \) and \( 1 - v_a(y) \geq t, t > v_a(y) \) which implies that \( v_a(x) < 0.5, v_a(y) < 0.5 \). Thus, \( 1 - v_a(x) > 0.5, 1 - v_a(y) > 0.5 \). Let \( a = (1 - v_a(x)) \lor (1 - v_a(y)) \), then \( a > 0.5 \). Therefore, \( A_a(x + y) \geq A_a(x) \land A_a(y) \geq 1/2 \lor 1/2 = 1/2 \), [Since \( 1 - v_a(x) \geq a, 1 - v_a(y) \geq a \). This implies that \( 1 - v_a(x + y) \geq a = (1 - v_a(x)) \lor (1 - v_a(y)) \geq s \lor t \). Therefore, \( \{x + y \mid A \} = 1/2 \).

(2) Similarly, we can prove that \( \{-x \mid A \} = 1/2 \).

(3) Let \( b = \{x \in \land qA \lor y \in \land qA\} \).

Case I. \( b = 1 \). Then, either \( \mu_a(x) \geq s, \mu_a(x) > -1 \) or \( \mu_a(y) \geq t, \mu_a(y) > -1 \). Therefore, \( \mu_a(x) > 0.5 \) or \( \mu_a(y) > 0.5 \). Let \( a = \mu_a(x) \lor \mu_a(y) \), then \( a > 0.5 \). Also, \( \mu_a(x) = a \) or \( \mu_a(y) = a \), and so \( A_a(x) = 1 \) or \( A_a(y) = 1 \). Thus, \( A_a(xy) = A_a(x) \lor A_a(y) = 1 \) which implies that \( A_a(xy) = 1 \), and so \( \mu_a(xy) \geq a = \mu_a(x) \lor \mu_a(y) \geq s \lor t \). Therefore, \( \{xy \mid A \} = 1 \).

Case II. \( b = 1/2 \). Then, either \( 1 - v_a(x) \geq s, s > v_a(x) \) or \( 1 - v_a(y) \geq t, t > v_a(y) \), which implies either \( v_a(x) < 0.5 \) or \( v_a(y) < 0.5 \). Thus, \( 1 - v_a(x) > 0.5 \) or \( 1 - v_a(y) > 0.5 \). Let \( a = (1 - v_a(x)) \lor (1 - v_a(y)) \), then \( a > 0.5 \). Therefore, \( A_a(xy) \geq A_a(x) \lor A_a(y) \geq 1/2 \lor 1/2 = 1/2 \), [Since \( 1 - v_a(x) = a \) or \( 1 - v_a(y) = a \). This implies that \( 1 - v_a(xy) \geq a = (1 - v_a(x)) \lor (1 - v_a(y)) \geq s \lor t \). Therefore, \( \{xy \mid A \} \geq 1/2 = \{x \in \land qA \lor y \in \land qA\} \).]

Theorem 3.13. An intuitionistic fuzzy set \( A = (\mu_A, \nu_A) \) of \( R \) is an intuitionistic fuzzy ideal with thresholds \((s,t)\) of \( R \) if and only if for any \( a \in (s,t] \), \( A_a \) is a fuzzy ideal of \( R \).

Proof. Suppose that \( A \) is an intuitionistic fuzzy ideal with thresholds \((s,t)\) of \( R \). Let \( a \in (s,t], x, y \in R \) and \( b = A_a(x) \lor A_a(y) \).

Case I. \( b = 1 \). Then, \( A_a(x) = 1 \) or \( A_a(y) = 1 \). This implies that \( \mu_a(x) \geq a > s \) or \( \mu_a(y) \geq a > s \). Now, \( \mu_a(xy) \lor s \geq (\mu_a(x) \lor \mu_a(y)) \lor a \geq (a \lor s) \lor t = a \). Therefore, \( \mu_a(xy) \geq a \) which implies that \( A_a(xy) = 1 \).

Case II. \( b = 1/2 \). Then, \( A_a(x) = 1/2 \) or \( A_a(y) = 1/2 \), which implies that \( 1 - v_a(x) \geq a \) or \( 1 - v_a(y) \geq a \). Thus, \( v_a(x) \land v_a(y) \leq 1 - a < 1 - s \). Now, \( v_a(xy) \land (1 - s) \leq (v_a(x) \land v_a(y)) \lor (1 - t) \leq (1 - a) \lor (1 - t) = 1 - a \), [Since \( t \geq a \) and \( s > 1 - a \). Therefore,
1 − v_A(xy) ≥ a, and so A_d(xy) ≥ 1/2 = A_d(x) ∨ A_d(y). Hence, A_d(xy) ≥ A_d(x) ∨ A_d(y).

Similarly, we have A_d(x + y) ≥ A_d(x) ∧ A_d(y) and A_d(−x) ≥ A_d(x).

Conversely, we assume for any a ∈ (s, t], A_a is a fuzzy ideal of R.

(1) To show μ_A(x + y) ∧ s ≥ μ_A(x) ∧ μ_A(y) ∧ t. If μ_A(x + y) ∧ s < a = μ_A(x) ∧ μ_A(y) ∧ t, then a ∈ (s, t] and μ_A(x) ∧ μ_A(y) ≥ a. Thus, from A_d(x + y) ≥ A_d(x) ∧ A_d(y) = 1, we have A_d(x + y) = 1, and so μ_A(x + y) ≥ a, which contradicts to μ_A(x + y) < a. Therefore, μ_A(x + y) ∧ s ≥ μ_A(x) ∧ μ_A(y) ∧ t.

(2) Similarly, we have μ_A(−x) ∧ s ≥ μ_A(x) ∧ t.

(3) To show μ_A(xy) ∨ s ≥ (μ_A(x) ∨ μ_A(y)) ∧ t. If μ_A(xy) ∨ s < a = (μ_A(x) ∨ μ_A(y)) ∧ t, then a ∈ (s, t] and μ_A(x) ≥ a or μ_A(y) ≥ a. Thus, from A_d(xy) ≥ A_d(x) ∨ A_d(y) = 1, we have A_d(xy) = 1, and so μ_A(xy) ≥ a, which contradicts to μ_A(xy) < a. Therefore, μ_A(xy) ∨ s ≥ (μ_A(x) ∨ μ_A(y)) ∧ t.

(4) To show v_A(x + y) ∧ (1 − s) ≤ (v_A(x) ∧ v_A(y)) ∧ (1 − t). If v_A(x + y) ∧ (1 − s) > a = (v_A(x) ∧ v_A(y)) ∧ (1 − t), then (1 − v_A(x + y)) ∧ s < b = 1 − a = (1 − v_A(x)) ∧ (1 − v_A(y)) ∧ t, and so b ∈ (s, t] and (1 − v_A(x)) ≥ b or (1 − v_A(y)) ≥ b. Thus, from A_d(x + y) ≥ A_d(x) ∧ A_d(y) = 1/2, we have A_d(x + y) ≥ 1/2, and so 1 − v_A(x + y) ≥ b = 1 − a. Therefore, v_A(x + y) ≤ a, which contradicts to v_A(x + y) > a. Hence, v_A(x + y) ∧ (1 − s) ≤ (v_A(x) ∧ v_A(y)) ∨ (1 − t).

(5) Similarly, we have v_A(−x) ∧ (1 − s) ≤ v_A(x) ∧ (1 − t).

(6) To show v_A(xy) ∧ (1 − s) ≤ (v_A(x) ∧ v_A(y)) ∨ (1 − t). If v_A(xy) ∧ (1 − s) > a = (v_A(x) ∧ v_A(y)) ∨ (1 − t), then (1 − v_A(xy)) ∧ s < b = 1 − a = (1 − v_A(x)) ∧ (1 − v_A(y)) ∧ t, and so b ∈ (s, t] and (1 − v_A(x)) ≥ b or (1 − v_A(y)) ≥ b. Thus, from A_d(xy) ≥ A_d(x) ∨ A_d(y) ≥ 1/2, we have A_d(xy) ≥ 1/2, and so 1 − v_A(xy) ≥ b = 1 − a. Therefore, v_A(xy) ≤ a, which contradicts to v_A(xy) > a. Hence, v_A(xy) ∧ (1 − s) ≤ (v_A(x) ∧ v_A(y)) ∨ (1 − t).

Hence, A = (μ_A, v_A) is an intuitionistic fuzzy ideal with thresholds (s, t) of R.

4. Conclusion

In this article, we have defined a new kind of fuzzy subring and ideal namely, (α, β)-intuitionistic fuzzy subrings and ideals, where α, β ∈ {∈, ∈ q, ∈ ∧ q, ∈ ∨ q}. Among the 16 such intuitionistic fuzzy ideals, (∈, ∈), (∈, ∈ ∨ q) and (∈ ∧ q, ∈) are significant. We have investigated various properties of (α, β)-intuitionistic fuzzy ideals and attempted to connect intuitionistic fuzzy ideal with thresholds (s, t). In our opinion this is an opening for investigations of different types of (α, β)-intuitionistic fuzzy ideals.

References


