Extending the structural homomorphism of LCC loops

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Abstract. A loop $Q$ is said to be left conjugacy closed if the set $A = \{L_x / x \in Q\}$ is closed under conjugation. Let $Q$ be an LCC loop, let $\mathcal{L}$ and $\mathcal{R}$ be the left and right multiplication groups of $Q$ respectively, and let $I(Q)$ be its inner mapping group, $M(Q)$ its multiplication group. By Drápal’s theorem [3, Theorem 2.8] there exists a homomorphism $\Lambda : \mathcal{L} \to I(Q)$ determined by $L_x \to R_x^{-1}L_x$. In this short note we examine different possible extensions of this $\Lambda$ and the uniqueness of these extensions.

Keywords: LCC loop, multiplication group, inner mapping group, homomorphism

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1. Introduction

$Q$ is a loop if it is a quasigroup with neutral element. The functions $L_a(x) = ax$ (left translation) and $R_a(x) = xa$ (right translation) are permutations on the elements of $Q$ for every $a \in Q$. The permutation group generated by left and right translations $M(Q) = \langle L_a, R_a / a \in Q \rangle$ is called the multiplication group of the loop $Q$. Denote $I(Q)$ the stabilizer of the neutral element in $M(Q)$. $I(Q)$ is a subgroup of $M(Q)$ and it is called the inner mapping group of $Q$. It is clear that $M(Q)$ is a transitive permutation group on the loop $Q$. Denote $A = \{L_a / a \in Q\}$ and $B = \{R_a / a \in Q\}$. It is well known that $A$ and $B$ are left (and right) transversals to $I(Q)$ in $M(Q)$ which satisfy $\langle A, B \rangle = M(Q)$, and the commutator subgroup $[A, B] \leq I(Q)$. Furthermore $\text{core}_{M(Q)} I(Q) = 1$ ($\text{core}_{M(Q)} I(Q)$ means the largest normal subgroup of $M(Q)$ in $I(Q)$).

The subgroups $\mathcal{L} = \langle L_a / a \in Q \rangle$ and $\mathcal{R} = \langle R_a / a \in Q \rangle$ are called left and right multiplication groups, respectively. Denote $\mathcal{L}_1 = \mathcal{L} \cap I(Q)$, $\mathcal{R}_1 = \mathcal{R} \cap I(Q)$ and $T_x = L_x^{-1}R_x$. A standard fact is that $I(Q)$ is generated by $\mathcal{L}_1 \cup \mathcal{R}_1 \cup \{T_x / x \in Q\}$.

The right nucleus of a loop $Q$ is

$$N_\rho = \{a \in Q / (xy)a = x(ya) \text{ for every } x, y \in Q\},$$

the left nucleus of a loop $Q$ is

$$N_\lambda = \{a \in Q / a(xy) = (ax)y \text{ for every } x, y \in Q\}.$$

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We have (see [3, Lemma 1.9])

\[ C_{M(Q)}(R) = \{ L_a / a \in N_\lambda \} \quad \text{and} \quad C_{M(Q)}(L) = \{ R_a / a \in N_\rho \}. \]

A subset \( A \) of a group \( G \) is said to be closed under conjugation if \( a_1a_2^{-1} \in A \) for all \( a_1, a_2 \in A \). This is clearly true if and only if \( A \) is a normal subset in \( \langle A \rangle \).

A loop \( Q \) is said to be conjugacy closed (CC) if the sets \( A = \{ L_x / x \in Q \} \) and \( B = \{ R_x / x \in Q \} \) are closed under conjugation. The concept of conjugacy closedness was introduced first by Soikis [7] and later independently by Goodaire and Robinson [4].

A loop \( Q \) is called left conjugacy closed (LCC) if the set \( A = \{ L_x / x \in Q \} \) is closed under conjugation. Thus for all \( a, b \in Q \) there exists \( c \in Q \) such that \( L_aL_bL_a^{-1} = L_c \). Hence in every LCC loop \( Q \) we have \( L_aL_bL_a^{-1} = L_{T_a(b)} \) for all \( a, b \in Q \).

LCC loops were also introduced by Soikis [7]. We have to mention Basarab’s paper [1], A. Drápal’s paper [3] and P. Nagy and K. Strambach’s paper [6]. This latter paper is in connection with geometry of LCC loops. As a Bol loop \( Q \) is LCC if and only if \( x^2 \in N_\lambda \) for all \( x \in Q \) we have to emphasize the relevance of the paper of G.P. Nagy with H. Kiechle [5].

A. Drápal studied in [2] the relationship within multiplication groups of conjugacy closed (CC) loops, and in his other paper [3] concerning LCC loops he could transfer some basic facts from CC loops to LCC loops. The following basic result — which has been used in proofs of many statements — can also be found in this latter paper [3]. This result was first obtained for CC loops in Drápal’s earlier paper [2].

**Theorem 1.1.** Let \( Q \) be a left conjugacy closed loop. Denote by \( L \) its left multiplication group. Then there exists a unique homomorphism: \( \Lambda : L \to I(Q) \) that maps \( L_x \) to \( T_x \) for each \( x \in Q \). This homomorphism is the identity on \( L_1 \) and its kernel is equal to \( Z(L) = \{ R_x / x \in Q \} \cap L \); furthermore if \( R_x \in Z(L) \), then \( x \in Z(N_\rho) \).

As the kernel of this homomorphism \( \Lambda \) does not contain the whole set \( \{ R_a / a \in N_\rho \} \) we cannot conclude — using this kernel — if the loop has non-trivial right nucleus. The purpose of this paper is to extend this homomorphism in such a way that the kernel consists of the set \( \{ R_a / a \in N_\rho \} \). Since \( C_{M(Q)}(L) = \{ R_a / a \in N_\rho \} \) it seems natural to examine the relationship between \( \{ g \in M(Q) / L_a^g \in A \} \) and \( \Lambda \). It turned out that we can really extend this \( \Lambda \) in the required way.

2. Extension

In this section, for the proofs of our theorems we need the following
Lemma 2.1. Let $Q$ be a loop, $M(Q)$ its multiplication group, $A = \{L_x / x \in Q\}$, $B = \{R_x / x \in Q\}$. Denote $H_0 = \{h \in I(Q) / A^h = A\}$. Then the following statements are true:

(i) $H_0 = I(Q) \cap \text{Aut}(Q)$;
(ii) $B^h = B$ for some $h \in I(Q)$ if and only if $h \in H_0$.

Proof: (i) First we show that if $h \in I(Q) \cap \text{Aut} Q$, then $h \in H_0$ i.e. $h L_a h^{-1} \in A$ for every $L_a \in A$.

Let $a_0 \in Q$ be arbitrary, and denote $h(a) = a^*$. Then $(h L_a h^{-1})(a_0) = h(ah^{-1}(a_0)) = h(a)a_0 = a^*a_0 = L_{a^*}(a_0)$. Consequently $h L_a h^{-1} = L_{a^*}$.

Conversely, let $h \in H_0$. It suffices to prove $h(xy) = h(x)h(y)$ for arbitrary $x, y \in Q$. Suppose $h L_x h^{-1} = L_{x_1}$, $h L_y h^{-1} = L_{y_1}$. Then

$h(x) = h(x \cdot 1) = (h L_x)(1) = (h L_x h^{-1})(1) = L_{x_1}(1) = x_1$ and

$h(y) = h(y \cdot 1) = (h L_y)(1) = (h L_y h^{-1})(1) = L_{y_1}(1) = y_1$.

Further $h(xy) = h(xy \cdot 1) = (h L_x L_y)(1) = (h L_x L_y h^{-1})(1) = L_{x_1} L_{y_1}(1) = x_1 y_1$.

(ii) By (i) it is obvious.

Lemma 2.2. Let $Q$ be an LCC loop, $M(Q)$ its multiplication group, $I(Q)$ its inner mapping group, $A = \{L_a / a \in Q\}$. Let $\Lambda$ be the homomorphism from Theorem 1.1. Suppose $h \in I(Q) \cap \text{Aut} Q$. Then $(\Lambda(L_a))^h = \Lambda(L_a^h)$ for every $L_a \in A$.

Proof: We have $L_a^h = L_{h^{-1}(a)}$, $T_a^h = T_{h^{-1}(a)}$. Using $\Lambda(L_a) = T_a$ we get our statement.

We observe that for every element $g$ of $M(Q)$ obviously both $(R_{g(1)})^{-1} g$ and $(L_{g(1)})^{-1} g$ belong to $I(Q)$.

In Drápal’s theorem (Theorem 1.1) this homomorphism $\Lambda$ maps $L_x$ to $T_x = R_{x}^{-1} L_x$ and it is the identity on $L_1$ (= $L \cap I(Q)$). Consequently, $\text{Im} \Lambda = \langle T_x / x \in Q \rangle$. As we have $L = A L_1$ and $A \cap L_1 = \{e\}$, this $\Lambda$ maps $L$ into $I(Q)$ in such a way that $\Lambda(\ell) = R_{\ell(1)}^{-1} \ell$ for every $\ell \in L$.

Extend this function $\Lambda$ to the whole $M(Q)$. Thus we consider the function $\Lambda_0$ which maps $g \in M(Q)$ to $R_{g(1)}^{-1} g \in I(Q)$.

Since $\Lambda_0$ is a homomorphism on $L$, the question arises: which is the largest subgroup of $M(Q)$ such that $\Lambda_0$ is a homomorphism on this subgroup. The following theorem gives the answer.

Theorem 2.3. Let $Q$ be an LCC loop. Let $\Lambda_0 : M(Q) \to I(Q)$ be such that $\Lambda_0(g) = R_{g(1)}^{-1} g$. Then the largest subgroup $L^*$ of $M(Q)$ such that the restriction of $\Lambda_0$ on $L^*$ is a homomorphism, is the following:

$L^* = \{g \in M(Q) / L_x^9 g \in A \text{ for every } L_x \in A\}$,

$L^* \cap I(Q) = I(Q) \cap \text{Aut} Q \quad \text{and} \quad L^* = L(I(Q) \cap \text{Aut} Q)$.  


Denote $\Lambda^*$ the restriction of $\Lambda_0$ on $\mathcal{L}^*$. Then $\Lambda^*$ is the identity on $\mathcal{L}^* \cap I(Q)$ and $\ker \Lambda^* = \{ R_x / x \in Q \} \cap \mathcal{L}^* = \{ R_a / a \in N_p \}$. Furthermore $\text{Im} \Lambda^*$ is generated by $(\mathcal{L}^* \cap I(Q)) \cup \{ T_x / x \in Q \}$.

**Proof:** The left conjugacy closedness implies $\mathcal{L}^* \supseteq \mathcal{L}$. Denote $\mathcal{U} = \{ g \in M(Q) / L_g^R \in A \text{ for every } L_x \in A \}$. Clearly $\mathcal{U}$ is a subgroup of $M(Q)$.

First we show $\mathcal{L}^* \subseteq \mathcal{U}$. Since $B \cap I(Q) = \{ e \}$ obviously $\Lambda_0$ is the identity on $I(Q)$. Let $h \in \mathcal{L}^* \cap I(Q)$, $L_a \in A$. Then $hL_a \in \mathcal{L}^*$ and $\Lambda_0(hL_a) = \Lambda_0(h)\Lambda_0(L_a) = hR_a^{-1}L_a$. On the other hand, $\Lambda_0(hL_a) = R_c^{-1}hL_a$ where $R_c^{-1}hL_a \in I(Q)$. Hence $R_c^{-1}h = R_c$ for every $a \in Q$, $h \in \mathcal{L}^* \cap I(Q)$, consequently $B^h = B$. Using Lemma 2.1 we obtain $\mathcal{L}^* \cap I(Q) \subseteq \mathcal{U}$. The left conjugacy closedness implies $\mathcal{L} \subseteq \mathcal{U}$.

We show $\Lambda_0$ is a homomorphism on $\mathcal{U}$. Let $\ell_1, \ell_2 \in \mathcal{U}$, clearly $\ell_1 = L_{a_1}h_1$, $\ell_2 = L_{a_2}h_2$ with $h_1, h_2 \in \mathcal{U} \cap I(Q)$. We prove $\Lambda_0(L_{a_1}h_1L_{a_2}h_2) = \Lambda_0(L_{a_1}h_1)\Lambda_0(L_{a_2}h_2)$. Clearly $\Lambda_0(L_{a_1}h_1) = R_a^{-1}L_{a_1}h_1 = \Lambda(L_{a_1})h_1$, $\Lambda_0(L_{a_2}h_2) = R_a^{-1}L_{a_2}h_2 = \Lambda(L_{a_2})h_2$. On the other hand, $\Lambda_0(L_{a_1}h_1L_{a_2}h_2) = \Lambda_0(L_{a_1}L_{a_2}^{-1}h_1h_2) = R_{d^{-1}}L_{a_1}L_{a_2}^{-1}h_1h_2$ for some $d \in Q$. By the definition of $\mathcal{U}$, $L_{a_2}^{-1}h_2 \in A$, whence $L_{a_1}L_{a_2}^{-1}h_1h_2 \in \mathcal{L}$, consequently $\Lambda(L_{a_1}L_{a_2}^{-1}h_1h_2) = h_1h_2$. Thus it suffices to show $\Lambda(L_{a_2}^{-1}h_2) = (\Lambda(L_{a_2}^{-1})h_1^{-1}$, but this follows immediately from Lemma 2.2.

Lemma 2.1 implies $\mathcal{L}^* \cap I(Q) = I(Q) \cap \text{Aut } Q$. Since $\mathcal{L} \subseteq \mathcal{L}^*$, we have $\mathcal{L}^* = \mathcal{L}(I(Q) \cap \text{Aut } Q)$.

As $\text{Im } \mathcal{L} = \langle T_x / x \in Q \rangle$ and $\mathcal{L}^*$ is the identity on $\mathcal{L}^* \cap I(Q)$ it follows $\text{Im } \Lambda^* = \{ f \in \mathcal{L}^* / \Lambda^*(f) = R_{f_1}^{-1}f \}$. Hence $f = R_{f_1}^{-1} \in B \cap \mathcal{L}^*$. From $[A, B] \leq I(Q)$ we get $L_x^{-1}L_{f_1}R_{f_1}^{-1} \in I(Q)$ for every $L_x \in A$, but $R_{f_1}^{-1} \in \mathcal{L}^*$ implies $L_{x^{-1}}R_{f_1}^{-1} \in A$, consequently $R_{f_1}^{-1} \in C_{M(Q)}(A)$, whence $R_{f_1}^{-1} \in C_{M(Q)}(A)$. Since $C_{M(Q)}(A) = \{ R_a / a \in N_p \}$, we conclude $\ker \Lambda^* = B \cap \mathcal{L}^* = \{ R_a / a \in N_p \}$.

Our next question is, whether every homomorphism of $\mathcal{L}^*$ into $I(Q)$ which coincides with $\Lambda$ on the elements of $\mathcal{L}$ is equal to $\Lambda^*$.

We give the answer:

**Theorem 2.4.** Let $Q$ be an LCC loop, $M(Q)$ its multiplication group, $I(Q)$ its inner mapping group, $A = \{ L_a / a \in Q \}$, $\mathcal{L}^* = \{ g \in M(Q) / L_g^R \in A \text{ for every } L_x \in A \}$.

Let $T = \langle T_x / x \in Q \rangle$, $c \in C_{I(Q)}(T)$. Define a function $\Lambda_1$ on $\mathcal{L}^*$: if $\ell = L_0h \in \mathcal{L}^*$ with $h \in \mathcal{L}^* \cap I(Q)$, then $\Lambda_1(\ell) = \Lambda(L_0)hc$. Then $\Lambda_1$ is a homomorphism on $\mathcal{L}^*$ and $\Lambda_1$ coincides with $\Lambda$ on the elements of $\mathcal{L}$. 


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PROOF: We have $\text{Im} \Lambda = \langle T_x \mid x \in Q \rangle$ and $L_1 \subseteq \text{Im} \Lambda$, whence $c \in C_G(L_1)$, consequently $\Lambda_1$ coincides with $\Lambda$ on the elements of $L$.

We show $\Lambda_1$ is a homomorphism on $L^*$. Let $\ell_1, \ell_2 \in L^*$, $\ell_1 = L_{a_1}h_1$, $\ell_2 = L_{a_2}h_2$ with $h_1, h_2 \in L^* \cap I(Q)$. By the definition of $\Lambda_1$, $\Lambda_1(\ell_1) = \Lambda(L_{a_1})h_1^c$, $\Lambda_1(\ell_2) = \Lambda(L_{a_2})h_2^c$. Clearly $\ell_1 \ell_2 = L_{a_1}L_{a_2}^{-1}h_1h_2$, since $(L_{a_2})^{-1} \in A$ we have $L_{a_1}(L_{a_2})^{-1} \in L$, whence $L_{a_1}(L_{a_2})^{-1} = L_{a_3}h_3$ with $h_3 \in L_1$. Consequently $\Lambda_1(\ell_1 \ell_2) = \Lambda(L_{a_3})(h_3h_1h_2)^c$. As $h_3 \in T$ it follows $\Lambda_1(\ell_1 \ell_2) = \Lambda(L_{a_3})h_3(h_1h_2)^c = \Lambda(L_{a_3}h_3)(h_1h_2)^c = \Lambda(L_{a_1})\Lambda(L_{a_2})h_1^{-1}(h_1h_2)^c$.

Thus it suffices to prove $(\Lambda(L_{a_2}))^{-1}(h_1^{-1})^c = \Lambda(L_{a_2}^{-1})$. Using $\text{Im} \Lambda = T$ and $c \in C_I(Q)(T)$ it is equivalent to $(\Lambda(L_{a_2}))^{-1}(h_1^{-1}) = \Lambda(L_{a_2}^{-1})$. The latter equality follows immediately from Lemma 2.2. □

Corollary 2.5. Let $Q$ be an LCC loop. Denote $T = \langle T_x \mid x \in Q \rangle$ and suppose that $C_I(Q)(T) \not\subseteq C_I(Q)(L^* \cap I(Q))$. Then there exists a homomorphism $\Lambda_1$ on $L^*$ which coincides with $\Lambda$ on $L$, but $\Lambda_1 \neq \Lambda^*$, where $\Lambda^*$ is the homomorphism described in Theorem 2.3.

References


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