On non-normality points and metrizable crowded spaces

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Abstract. $\beta X - \{p\}$ is non-normal for any metrizable crowded space $X$ and an arbitrary point $p \in X^*$.

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1. Introduction

We investigate non-normality points in Čech-Stone remainders $X^* = \beta X - X$ of metrizable spaces.

There are several simple proofs that, under CH, $\omega^* - \{p\}$ is not normal for any $p \in \omega^*$ [7], [8]. “Naively” it is known only for special points of $\omega^*$. If $p$ is an accumulation point of some countable discrete subset of $\omega^*$, or if $p$ is a strong $R$-point, or if $p$ is a Kunen’s point, then $\omega^* - \{p\}$ is not normal (Blaszczyk and Szymanski [1], Gryzlov [2], van Douwen respectively).

What about realcompact crowded spaces? Is $\beta X - \{p\}$ non-normal whenever $X$ is realcompact and crowded and $p \in X^*$? Probably, but we are unaware of any counterexample. On the other hand, the answer is “yes” if $X$ is a locally compact Lindelöf separable crowded space with $\pi w(X) \leq \omega_1$ and $p$ is remote [5]. It is also “yes” if $X$ is a second countable crowded space and either $X$ is locally compact, or $X$ is zero-dimensional, or $p$ is remote [3], [4], [6]. Using the regular base of Arhangel’skiǐ J. Terasawa has omitted the separability condition in the last two cases. He has obtained the affirmative answer in case if $X$ is a metrizable crowded space and either $X$ is strongly zero-dimensional or $p$ is remote [10]. Here, introducing $p$-filters into this construction, we answer affirmatively for all metrizable crowded spaces.

B. Shapirovskij [9] has defined a butterfly-point (or $b$-point) in a space $X$. We call $p \in X^*$ a butterfly-point in $\beta X$, if $\{p\} = \text{Cl } F \cap \text{Cl } G$ for some $F, G \subset X^* - \{p\}$ with $\text{Cl } (F \cup G) \subset X^*$.

Theorem. Let $X$ be a non-compact metrizable crowded space. Then any point $p \in X^*$ is a butterfly-point in $\beta X$. Hence $\beta X - \{p\}$ is not normal.
2. Proofs

From now on a space $X$ is non-compact, metrizable and crowded, i.e. $X$ has no isolated points, and $p \in X^*$ is an arbitrary point. We denote by cl- and Cl-
the closure operations in $X$ and $\beta X$ respectively, $3 = \{0, 1, 2\}$.

Let $\pi$ and $\sigma$ be an arbitrary families. A set $U \in \pi$ is called a maximal member of the family $\pi$ if $U \subsetneq V$ for no $V \in \pi$. If members of $\pi$ are mutually disjoint (with closure), then $\pi$ is called (strongly) cellular. We write $\pi \prec \sigma$ if $U \cap V \neq \emptyset$ implies $U \supsetneq V$ for any $U \in \pi$ and $V \in \sigma$. We denote by $\text{Exp} \pi$ the set of subfamilies $\{F : F \subset \pi\}$. We define a projection $f_{\sigma}^\pi$ from $\text{Exp} \pi$ to $\text{Exp} \sigma$ by $f_{\sigma}^\pi F = \{V \in \sigma : \bigcup F \cap V \neq \emptyset\}$ for every $F \in \text{Exp} \pi$.

A maximal locally finite cellular family of open sets is called nice. The introduced in [6] cellular refinement $\text{Cel} (\pi) = \{\bigcap \phi - \text{cl} (\bigcup (\pi - \phi) : \phi \subset \pi\}$ of $\pi$ is nice, if $\pi$ is an open locally finite cover of $X$.

Let $\pi$ and $\sigma$ be nice families. A collection $\mathcal{F} = \{F\}$ of subfamilies $F \subset \pi$ is called a $p$-filter on $\pi$, if $p \in \text{Cl} \bigcup_{k=0}^{n} F_k$ for any finite subcollection $\{F_0, \ldots, F_n\} \subset \mathcal{F}$. Obviously, the union of any increasing family of $p$-filters is also a $p$-filter. So by Zorn's lemma there are maximal $p$-filters or $p$-ultrafilters $\mathcal{F}'$ on $\pi$, that is $\mathcal{F}' = \mathcal{G}$ for any $p$-filter $\mathcal{G}$ with $\mathcal{F}' \subset \mathcal{G}$. Adding step-by-step new subfamilies from $\text{Exp} \pi - \mathcal{F}$ to $\mathcal{F}$, while possible, we can embed any $p$-filter $\mathcal{F}$ into some $p$-ultrafilter $\mathcal{F}'$. If $p$ is not a remote point, distinct $p$-ultrafilters $\mathcal{F}'$ may exist. But each of them contains $\pi(O) = \{V \in \pi : V \cap O \neq \emptyset\}$ for any neighborhood $O$ of $p$ and its image $f_{\sigma}^\pi F = \{f_{\sigma}^\pi F : F \in \mathcal{F}\}$ is a $p$-filter on $\sigma$. We write $\pi \prec_{\mathcal{F}} \sigma$, if there is $F \in \mathcal{F}$ with $F \prec \sigma$. We denote $\bigcap \mathcal{F}^* = \bigcap \{\text{cl} \bigcup F : F \in \mathcal{F}\}$.

For every $i \in \mathbb{N}$ we fix an open locally finite cover $\mathcal{P}_i$ of $X$ so that $\text{diam} U \leq \frac{1}{i}$ for any $U \in \mathcal{P}_i$ and $\{V \in \mathcal{P}_j : V \cap U \neq \emptyset\}$ is finite for each $j < i$. Then it is easy to see that

$$\mathcal{P} = \bigcup_{i \in \mathbb{N}} \mathcal{P}_i$$

is a regular base of Arhangel'skiĭ, i.e. for any point $x \in X$ and for any its neighborhood $O \subset X$ there is another neighborhood $O' \subset X$ of $x$ with the following properties: $O' \subset O$ and at most finitely many members of $\mathcal{P}$ meet both $O'$ and $X - O$ simultaneously. Moreover, for any cover $\pi \subset \mathcal{P}$ the family of its maximal members is a locally finite subcover of $X$.

By induction (see, also, [6]) we define the families of non-empty open sets $\mathcal{D}_k$ and $\mathcal{W}_k \subset \mathcal{P}$ for all $k \in \mathbb{N}$ as follows:

$$\mathcal{D}_1 = \text{Cel} (\mathcal{P}_1).$$

If a nice family $\mathcal{D}_k = \{U\}$ has been constructed, then

$$\mathcal{W}_k = \{U(\nu) : U \in \mathcal{D}_k \text{ and } \nu \in \{3\}\}$$
is strongly cellular with \( \text{cl } U(\nu) \subset U \) for any its member and

\[
D_{k+1} = \text{Cel}(D_k \cup W_k \cup P_{k+1}).
\]

By our construction, if \( U, V \in \bigcup_{k \in \mathbb{N}} D_k \) are not disjoint, then either \( U \subseteq V \) or \( U \supseteq V \). For any \( U \in P_k \) the family \( \hat{U} = \{V \in D_k : V \cap U \neq \emptyset\} \) is locally finite and nice in \( U \). For any locally finite cover \( \pi \subset \mathcal{P} \) we denote \( \sigma(\pi) \) all maximal members of the family \( \bigcup \{\hat{U} : U \in \pi\} \). Then \( \sigma(\pi) \) is nice. Define

\[
\Sigma = \{ \sigma(\pi) : \pi \subset \mathcal{P} \text{ is a locally finite cover of } X \}
\]

and put \( \sigma(\nu) = \{U(\nu) : U \in \sigma\} \) for any \( \sigma \in \Sigma \) and \( \nu \in 3 \).

**Lemma 1.** If \( \pi \) is an open locally finite cover of \( X \), then \( \text{Cel}(\pi) \) is nice.

**Proof:** Let \( \phi \subset \pi \). If \( \bigcap \phi \neq \emptyset \), then \( \phi \) is finite. So \( \bigcap \phi \) and, hence, \( \bigcap \phi - \text{cl}(\pi - \phi) \) is open.

Let \( \phi, \phi' \subset \pi \) be different and \( U \in \phi - \phi' \). Then \( \bigcap \phi \subset U \) and \( \bigcap \phi' \cap U = \emptyset \), because \( U \in \pi - \phi' \).

Let a neighborhood \( O \) of \( x \in X \) meet finitely many members of \( \pi \), say \( U_1, \ldots, U_k \). If \( \phi \subset \pi \) contains some \( U \in \pi - \{U_1, \ldots, U_k\} \), then \( \bigcap \phi \subset U \subset X - O \). So \( O \) meets at most \( 2^k \) members of \( \text{Cel}(\pi) \).

As \( \pi \) is a locally finite family of open sets, \( K = \bigcup \{\text{cl } U - U : U \in \pi\} \) is nowhere dense. Let \( x \notin K \) and \( \phi = \{U \in \pi : x \in U\} \). Then \( U \notin \phi \) implies \( x \notin \text{cl } U \). So \( x \in \bigcap \phi - \text{cl} \bigcup (\pi - \phi) \), because \( \pi \) is conservative, and \( \text{Cel}(\pi) \) is maximal. Our proof is complete. \( \square \)

**Lemma 2.** There is a well-ordered chain \( \{\sigma_\alpha : \alpha < \lambda\} \subset \Sigma \) and \( p \)-ultrafilters \( \mathcal{F}_\alpha \) on \( \sigma_\alpha \) with the following properties for all \( \alpha < \beta < \lambda \) and \( f_\beta^\alpha = f_\sigma^\beta_\alpha \):

1. \( p \notin \text{cl } U \) for each \( U \in \sigma_0 \);
2. \( f_\beta^\alpha \mathcal{F}_\alpha \subset \mathcal{F}_\beta \);
3. \( \sigma_\alpha \prec F_\alpha \sigma_\beta \);
4. for any \( \sigma \in \Sigma - \{\sigma_\alpha : \alpha < \lambda\} \) there is \( \alpha < \lambda \) with \( \lnot (\sigma_\alpha \prec F_\alpha \sigma) \).

**Proof:** Let \( \pi \) be all maximal members of the cover \( \{U \in \mathcal{P} : p \notin \text{cl } U\} \) and let \( \mathcal{F}_0 \) be any \( p \)-ultrafilter on \( \sigma_0 = \sigma(\pi) \).

For any ordinal \( \beta \) assume \( p \)-ultrafilters \( \mathcal{F}_\alpha \) on \( \sigma_\alpha \in \Sigma \) have been constructed for all \( \alpha < \beta \). If some \( \sigma \in \Sigma - \{\sigma_\alpha : \alpha < \beta\} \) satisfies the condition \( \sigma_\alpha \prec F_\alpha \sigma \) for all \( \alpha < \beta \), then we put \( \sigma_\beta = \sigma \) and embed the \( p \)-filter \( \bigcup_{\alpha<\beta} f_\beta^\alpha \mathcal{F}_\alpha \) into some \( p \)-ultrafilter \( \mathcal{F}_\beta \) on \( \sigma_\beta \). Otherwise our construction is complete. \( \square \)

**Lemma 3.** \( \bigcap \mathcal{F}_0^* \subset X^* \).

**Proof:** Let \( x \in X \) be an arbitrary point. Then \( F = \{U \in \sigma_0 : x \notin \text{cl } U\} \) satisfies, obviously, \( x \notin \text{cl } \bigcup F \) and \( F \in \mathcal{F}_0 \). \( \square \)
Lemma 4. If \( \alpha < \beta < \lambda \), then \( \bigcap F^*_\beta \subset \bigcap F^*_\alpha \).

Proof: There is \( F \in F_\alpha \) with \( F \prec \sigma_\beta \) by (3). For any \( G \in F_\alpha \) we have \( G \cap F \in F_\alpha \) and \( G \cap F \prec \sigma_\beta \). But then
\[
\bigcap F^*_\beta \subset \text{Cl} f^*_\beta (G \cap F) \subset \text{Cl} (G \cap F) \subset \text{Cl} G.
\]
\end{proof}

Lemma 5. For any neighbourhood \( O \) of \( p \) in \( \beta X \) there is \( \alpha < \lambda \) with \( \bigcap F^*_\alpha \subset O \).

Proof: Let \( \text{Cl} O' \subset O \) for a neighbourhood \( O' \) of \( p \) and let \( \pi \) be all maximal members of the cover \( \{U \in \mathcal{P} : U \cap O' \neq \emptyset \Rightarrow U \subset O\} \). For \( \sigma = \sigma(\pi) \) there is \( \alpha < \lambda \) with \( - (\sigma_\alpha \prec F_\alpha \sigma) \) by (3) or (4). As \( \sigma_\alpha (O') \in F_\alpha \) then \( F = \{V \in \sigma_\alpha (O') : V \subset U \text{ for some } U \in \sigma\} \) also belongs \( F_\alpha \). So \( \bigcap F^*_\alpha \subset \text{Cl} \bigcup F \subset \text{Cl} \bigcup \sigma (O') \subset \text{Cl} O \).
\end{proof}

Proposition 6. For any \( \alpha < \lambda \) and \( \nu \in 3 \) there is a point \( p_\alpha (\nu) \in \bigcap F^*_\alpha \) such that \( p_\alpha (\nu) \in \text{Cl} \bigcup \sigma_\beta (\nu) \) for all \( \beta \in \lambda - \alpha \).

Proof: Let \( \alpha < \beta_0 < \ldots < \beta_n < \lambda \) be any finite sequence and \( F \in F_\alpha \). Our idea is to find non-empty \( W \in \bigcup_{i \leq n} \sigma_\beta_i \) so that
\[
W(\nu) \subseteq \bigcap \bigcup_{i \leq n} \sigma_\beta_i (\nu) \cap \bigcup F.
\]

At the first step of induction we put \( \Delta_0 = \{\sigma_\beta_i : i \leq n\} \), \( \Theta_0 = \emptyset \) and choose \( W_0 \in \bigcup \Delta_0 \) as follows: We may assume \( F \prec \sigma_\beta_0 \). For any \( i < n \) there is \( G_i \in F_\beta_i \) with \( G_i \prec \sigma_\beta_{i+1} \). We denote \( F_0 = f^\alpha_\beta_0 F \cap G_0 \) and \( F_{i+1} = f^\beta_{i+1} F_i \cap G_{i+1} \). Then \( F_{i+1} \succ F_i \) and \( \bigcup F_{i+1} \subseteq \bigcup F_i \). Any pairwise intersecting \( U_i \in F_i \) make up an embedded sequence \( U_n \subseteq \ldots \subseteq U_0 \subseteq \bigcup F \). We define \( W_0 = U_0 \).

For any \( m < n \) let \( \Delta_m, \Theta_m \subset \Delta_0 \) and \( W_m \in \bigcup \Delta_m \) has been constructed so that
\begin{enumerate}
\item \( \Delta_m \cap \Theta_m = \emptyset \);
\item \( \Delta_m \cup \Theta_m = \Delta_0 \);
\item \( W_m \subseteq \bigcup F \);
\item \( W_m \subseteq \bigcup \sigma (\nu) \) for any \( \sigma \in \Theta_m \);
\item for any \( \sigma \in \Delta_m \) there is \( U_\sigma \in \sigma \) with \( U_\sigma \subseteq W_m \).
\end{enumerate}

Let \( \Omega_m = \{\sigma \in \Delta_m : U_\sigma = W_m\} \).

If \( \Delta_m \neq \Omega_m \), then we put \( \Delta_{m+1} = \Delta_m - \Omega_m \) and \( \Theta_{m+1} = \Theta_m \cup \Omega_m \). As \( \sigma \in \Delta_{m+1} \) are nice, we can choose \( U'_\sigma \in \sigma \) so that \( \bigcap \{U'_\sigma \in \sigma : \sigma \in \Delta_{m+1}\} \cap W_m (\nu) \neq \emptyset \). Then \( U_\sigma \subseteq W_m \) implies \( U'_\sigma \subseteq W_m (\nu) \) by our construction. We define \( W_{m+1} \) to be the maximal member of embedded sequence \( \{U'_\sigma : \sigma \in \Delta_{m+1}\} \).

If, finally, \( \Delta_m = \Omega_m \), then \( W_m \) is as required.
\end{proof}
PROOF OF THEOREM: Define \( F_\nu = \{ p_\alpha(\nu) : \alpha < \lambda \} \) for all \( \nu \in 3 \). By our construction, \( F_\nu \subset \bigcap F^*_0 \subset X^* \) and for any neighbourhood \( O \) of \( p \) there is \( \alpha < \lambda \) with \( \{ p_\beta(\nu) : \beta \in \lambda - \alpha \} \subset \bigcap F^*_\alpha \subset O \). Then the condition \( \{ p_\beta(\nu) : \beta < \alpha \} \subset \text{Cl} \bigcup \sigma_\alpha(\nu) \) implies that the sets \( \text{Cl} F_\nu - \{ p \} \) are pairwise disjoint and \( p \in F_\nu \) for no more then one unique \( F_\nu \). The other two ensure that \( p \) is a \( b \)-point in \( \beta X \).

Our proof is complete. \[ \square \]

REFERENCES


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