Further remarks on the Nemitskii operator in Hölder spaces

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Abstract. The paper is concerned with the Nemitskii operator in Hölder spaces. Namely conditions are given to ensure acting, continuity, Lipschitz and differentiability properties.

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0. Introduction.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the usual norm denoted by $| \cdot |$. In what follows $\Omega$ will denote an open bounded subset of $\mathbb{R}^n$ unless otherwise stated and $\overline{\Omega}$ its closure.

For $\alpha \in (0, 1]$, $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ is the space of all real functions $u$ which are $\alpha$-Hölder continuous in $\overline{\Omega}$, i.e. are such that: $h_\alpha(u) := \sup\{|u(x) - u(y)|/|x - y|^\alpha, \ x, y \in \overline{\Omega}, \ x \neq y\} < \infty$. $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ is a Banach space with the norm: $\|u\|_\alpha = \|u\|_\infty + h_\alpha(u)$ where $\|u\|_\infty = \sup\{|u(x)|; \ x \in \overline{\Omega}\}$.

This paper is concerned with the study in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ of some properties of the so called Nemitskii operator, i.e. the operator $F(u)(x) = f(x, u(x)), \ x \in \overline{\Omega}$ where $f = f(x, u)$ is a real valued function defined on $\overline{\Omega} \times \mathbb{R}$.

This argument has been deeply studied mainly in eastern Europe (see [1] and [2] for a complete bibliography). Among the others we like to mention P. Drábek [4] who has found necessary and sufficient conditions for $f = f(u)$ to induce a continuous Nemitskii operator mapping $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself.

Theorem 1.1 is simply a translation in words of [2, Theorem 7.3]; Theorem 3.1 extends the analogue in [2] which deals only with the case $f = f(u)$, as Theorems 2.1 and 4.1 do in relation with the ones in [5]. Finally Theorems 1.1, 2.1 and 4.1 extend our previous paper [7] since the actual assumptions are sensibly weaker.

We have now to compare our paper with the very recent one by M. Goebel [6]. First, we prove most of our results for any open bounded $\Omega \subset \mathbb{R}^n$ rather than for $\Omega = (a, b)$ as in [6]. (The extension to the case $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ is straightforward, see our final remark.)

Also, in [6] only sufficient conditions on $f$ are given so that $F$ has the various desired properties in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$, while we prove also some necessary conditions (Theorems 2.2 and 3.1) which in particular — in case $\Omega = (a, b)$ — yield a characterization of the local Lipschitz property of $F$ (Corollary 3.2).
Let us next discuss the conditions given here with those in [6]. To see this in some
detail, we state here two basic assumptions — for a given function \( g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) — to be used through the paper:

\[
g = g(x, u) \text{ is continuous in } \overline{\Omega} \times \mathbb{R} \\
\text{and } \alpha\text{-Hölder continuous in } x,
\]
uniformly with respect to \( u \) in compact intervals of \( \mathbb{R} \).

\[
g = g(x, u) \text{ is } \alpha\text{-Hölder continuous in } x,
\]
unifor\mly with respect to \( u \) in compact intervals of \( \mathbb{R} \),

\[
\text{and locally Lipschitz continuous in } u,
\]
uniformly with respect to \( x \in \overline{\Omega} \).

It is quite clear (see also the proof of Theorem 1.1) that (H) is a weaker assump-
tion than (K).

We note that (H) is equivalent to the assumption that \( g \) be continuous and satisfy
(A) of [6], while (K) is the same as (B) of [6].

As remarked in [6], if \( f \) satisfies (A) and is differentiable with respect to \( u \) with
\( f'_u \) continuous, then \( f \) satisfies (B) = (K). On the basis of this remark, it is easy
to check that the various properties of \( F \) (acting, continuity, etc.) are established
in our paper under conditions on \( f \) that are weaker than those in [6]. In particular,
we note that requiring existence and continuity of \( f'_u \) in order to prove the acting
property of \( F \) is an unnecessarily strong assumption (compare Theorem 1.1 with [6,
Theorem 1]). Theorem 2.1 and especially Theorem 2.2 below show that existence
of \( f'_u \) should be required at the level of continuity of \( F \).

We should finally mention that our proofs are sensibly different from those in [6],
and in particular the proof of Theorem 4.1 (differentiability) seems to us simpler
and more transparent.

1. Acting property.

**Theorem 1.1.** In order that the Nemitskii operator \( F \) generated by \( f \) map
\( C^{0,\alpha}(\overline{\Omega}, \mathbb{R}) \) into itself and be bounded, it is sufficient that \( f \) satisfies the assump-
tion (K). If \( \Omega = (a, b) \), this condition is also necessary.

**Proof:** By Theorem 7.3 in [2] it is sufficient to prove that (K) is equivalent to:

\[
\forall R > 0 \, \exists M > 0 : \\
\forall u, v \in [a, b] : \\
|f(x, u) - f(y, v)| \leq M \left\{ |x - y|^{\alpha} + \frac{|u - v|}{R} \right\} \\
\text{with } |u|, |v| \leq R, \forall x, y \in \overline{\Omega}.
\]

Indeed if (1.1) holds, then \( f \) is \( \alpha \)-Hölder in \( x \) since if \( R > 0 \), \( |u| \leq R \), and \( x, y \in \overline{\Omega} \),
then \( |f(x, u) - f(y, v)| \leq M |x - y|^{\alpha} \). Moreover (1.1) implies that \( f \) is locally Lipschitz
in \( u \) since, given \( R > 0 \), \( \exists M > 0 : |f(x, u) - f(x, v)| \leq M \frac{|u - v|}{R}, \forall |u|, |v| \leq R, \)
Assume now that $f$ satisfies (K); let $R > 0$, and let $L$ be the Lipschitz constant of $f$ in $[-R, R]$ and $k$ its Hölder constant in $\Omega$. We get:

$$|f(x, u) - f(y, v)| \leq |f(x, u) - f(x, v)| + |f(x, v) - f(y, v)|$$

$$\leq L|u - v| + k|x - y|^\alpha \quad (|u|, |v| \leq R, x, y \in \Omega)$$

and this yields (1.1) with $M = \max(LR, k)$. □

2. Continuity.

**Theorem 2.1.** Let $f$ satisfy the assumption (K) (so that $F$ acts in $C^{0,\alpha}(\Omega, \mathbb{R})$). If moreover $f$ is differentiable with respect to $u$ and $f'_u$ satisfies the assumption (H), then $F$ is continuous.

**Proof:** Let $u, v \in C^{0,\alpha}(\Omega, \mathbb{R})$. To estimate $h_\alpha(F(u + v) - F(u))$, we write (for $x, y \in \Omega$)

$$w(x, y) \equiv f(x, u(x) + v(x)) - f(x, u(x)) - f(y, u(y) + v(y)) + f(y, u(y))$$

$$= f(x, u(x) + v(x)) - f(x, u(y) + v(y)) + f(x, u(y) + v(y)) - f(x, u(x))$$

$$- f(y, u(y) + v(y)) + f(y, u(x)) - f(y, u(x)) + f(y, u(y))$$

$$= (u(x) + v(x) - u(y) - v(y)) \int_0^1 f'_u(x, u(y) + v(y)) +$$

$$\tau(u(x) + v(x) - u(y) - v(y))) d\tau$$

$$- (u(x) - u(y) - v(y)) \int_0^1 f'_u(x, u(y) + v(y) + \tau(u(x) - u(y) - v(y))) d\tau$$

$$+ (u(x) - u(y) - v(y)) \int_0^1 f'_u(y, u(y) + v(y) + \tau(u(x) - u(y) - v(y))) d\tau$$

$$- (u(x) - u(y)) \int_0^1 f'_u(y, u(y) + \tau(u(x) - u(y))) d\tau$$

$$= (u(x) - u(y)) \int_0^1 \left\{ f'_u(x, u(y) + v(y) + \tau(u(x) + v(x) - u(y) - v(y)))$$

$$- f'_u(x, u(y) + v(y) + \tau(u(x) - u(y) - v(y)))$$

$$+ f'_u(y, u(y) + v(y) + \tau(u(x) - u(y) - v(y)))$$

$$- f'_u(y, u(y) + \tau(u(x) - u(y)))) \right\} d\tau$$

$$+ (v(x) - v(y)) \int_0^1 f'_u(x, u(y) + v(y) + \tau(u(x) + v(x) - u(y) - v(y))) d\tau$$

$$+ v(y) \int_0^1 \left\{ f'_u(x, u(y) + v(y) + \tau(u(x) - u(y) - v(y)))$$

$$- f'_u(y, u(y) + v(y) + \tau(u(x) - u(y) - v(y))) \right\} d\tau.$$
Let now $\varepsilon > 0$ be given; set $M = \|u\|_\alpha$, $R = M + 1$. Since $f'_u$ is uniformly continuous in $\overline{\Omega} \times [-2R, 2R]$, then:

(a) there exists a constant $N$ such that $N = \max\{|f'_u(x,u)| : x \in \overline{\Omega}, u \in [-2R, 2R]\}$, 

(b) $\forall \varepsilon' > 0 \exists \delta'$ such that: $|f(x,u) - f(x,v)| < \varepsilon'$ whenever $x \in \overline{\Omega}$, $u,v \in [-2R, 2R]$ and $|u - v| < \delta'$.

Moreover $f'_u$ is $\alpha$-Hölder in $x$, namely there exists a non negative constant $L$ such that: $|f'_u(x,u) - f'_u(y,u)| \leq L|x - y|^\alpha$ for any $x,y \in \overline{\Omega}$, and $u \in [-2R, 2R]$. Then, if $\varepsilon' = \varepsilon/2M$ and $\delta = \min\{\delta', 1, \frac{\varepsilon}{N}, \frac{\varepsilon}{N}\}$ one gets, if $\|v\|_\alpha < \delta$:

$$|w(x,y)| \leq 4\varepsilon|x - y|^\alpha \quad (x,y \in \overline{\Omega})$$

whence $h_\alpha(F(u + v) - F(u)) \leq 4\varepsilon$.

To conclude, note that $f(x,u(x) + v(x)) - f(x,u(x)) = \int_0^1 f'_u(x,u(x) + \tau v(x))v(x) d\tau$ and hence $\|F(u + v) - F(u)\|_\infty \leq N\|v\|_\alpha < \varepsilon$. \hfill $\square$

**Theorem 2.2.** Let $f$ satisfy the assumption (K). If $F$ is continuous, then $f$ is differentiable with respect to $u$.

**Proof:** Since $f$ is $\alpha$-Hölder continuous in $x$ and locally lipschitzian in $u$ by Theorem 1.1, then $f$ is absolutely continuous in $u$ and hence almost everywhere differentiable with respect to $u$ in $\mathbb{R}$ in the following sense: for every $x \in \Omega$ the set $N_x = \{u : f'_u(x,u) \text{ does not exist}\}$ has zero Lebesgue measure in $\mathbb{R}$. It follows that its complement $N^c_x$ is dense in $\mathbb{R}$. We want to prove that $N^c_x = \mathbb{R}$ for every $x$.

Let us proceed by contradiction. Assume $N_{x_0} \neq \emptyset$ for some $x_0 \in \Omega$ and let $u_0 \in N_{x_0}$; thus setting

$$l_1 = \liminf_{h \to 0} \frac{f(x_0,u_0 + h) - f(x_0,u_0)}{h}, \quad l_2 = \limsup_{h \to 0} \frac{f(x_0,u_0 + h) - f(x_0,u_0)}{h}$$

we should have $l_1 < l_2$. Let $h_n$ and $\chi_n$ be real sequences converging to zero such that:

$$l_1 = \lim_{n \to \infty} \frac{f(x_0,u_0 + \chi_n) - f(x_0,u_0)}{\chi_n}, \quad l_2 = \lim_{n \to \infty} \frac{f(x_0,u_0 + h_n) - f(x_0,u_0)}{h_n}$$

and let $y_n$ and $x_n$ be sequences in $\Omega$ such that $h_n = |y_n - x_0|^\alpha$ and $\chi_n = |x_n - x_0|^\alpha$ (take e.g. $y_n = x_0 + h_n^{-1}v$, $|v| = 1$); then $x_n$ and $y_n$ both converge to $x_0$. By the density of $N^c_{x_0}$ there exists a real sequence $\theta_m$ converging to zero such that $f'_u(x_0,u_0 + \theta_m)$ exists for any $m$ and

$$f'_u(x_0,u_0 + \theta_m) = \lim_{\xi \to 0} \frac{f(x_0,u_0 + \xi + \theta_m) - f(x_0,u_0 + \theta_m)}{\xi} \quad (m \in \mathbb{N}).$$
Hence also:

\[
f_u'(x_0, u_0 + \theta_m) = \lim_{n \to \infty} \frac{f(x_0, u_0 + h_n + \theta_m) - f(x_0, u_0 + \theta_m)}{h_n} = \lim_{n \to \infty} \frac{f(x_0, u_0 + \chi_n + \theta_m) - f(x_0, u_0 + \theta_m)}{\chi_n}.
\]

We will prove that \( l_2 = \lim_{m \to \infty} f_u'(x_0, u_0 + \theta_m). \)

Let \( y_n \) be defined as above and consider, for any \( n, m \), the following expression:

\[
|h_n^{-1}[f(x_0, u_0 + h_n + \theta_m) - f(x_0, u_0 + \theta_m)] - f(y_n, u_0 + h_n) + f(x_0, u_0) - f(y_n, u_0 + h_n + \theta_m) + f(y_n, u_0 + h_n + \theta_m)]|.
\]

If we define \( u(x) = |x - x_0|^\alpha + u_0 \), so that \( u(y_n) = h_n + u_0 \) and \( u(x_0) = u_0 \), the expression in (2.1) is less than or equal to

\[
\|F(u + \theta_m) - F(u)\|_\alpha + \|F(u_0 + h_n) - F(u_0 + h_n + \theta_m)\|_\alpha.
\]

Letting \( n \to \infty \) and using the continuity of \( F \) in \( u_0 + \theta_m \) we get for any \( m \):

\[
|l_2 - f_u'(x_0, u_0 + \theta_m)| \leq \|F(u + \theta_m) - F(u)\|_\alpha + \|F(u_0) - F(u_0 + \theta_m)\|_\alpha.
\]

Letting now \( m \to \infty \) we get \( l_2 = \lim_{m \to \infty} f_u'(x_0, u_0 + \theta_m) \). The same argument shows that \( l_1 = \lim_{m \to \infty} f_u'(x_0, u_0 + \theta_m) \), so that \( l_1 = l_2 \): contradiction.  

**Corollary 2.3.** Let \( \Omega = (a, b) \) and assume that the Nemitskii operator \( F \) induced by \( f \) acts in \( C^{0, \alpha}(\Omega, \mathbb{R}) \) is bounded and continuous. Then \( f \) is differentiable with respect to \( u \).

### 3. Lipschitz property.

**Theorem 3.1.** Let \( f \) satisfy the assumption (K). In order that \( F \) be locally lipschitzian, it is sufficient that \( f \) be differentiable with respect to \( u \) and \( f_u' \) satisfy the assumption (K). If \( \Omega = (a, b) \), this condition is also necessary.

**Proof:** The “if” part can be proved in the same way as [7, Theorem 1.2].

To prove the “only if” part, note that by assumption

\[
\forall R > 0 \exists k(R) \geq 0 : \quad \|F(u) - F(v)\|_\alpha \leq k(R)\|u - v\|_\alpha \quad \forall \|u\|_\alpha, \|v\|_\alpha \leq R.
\]

Let \( u \in C^{0, \alpha}(\Omega, \mathbb{R}) \) with \( \|u\|_\alpha = M, R = M + 1 \) and \( \lambda \in (0, 1) \), so that \( \|u + \lambda\|_\alpha < R \).

Let us consider, for any \( x \in [a, b] \), the function: \( g(x, \lambda) = \lambda^{-1}[f(x, u(x) + \lambda) - f(x, u(x))] \). As a consequence of (3.1) the function \( g \) has the following properties:

(i) \( |g(x, \lambda) - g(y, \lambda)| \leq k(R)|x - y|^\alpha \quad (x, y \in [a, b], \lambda \in (0, 1)) \)

(ii) \( |g(x, \lambda)| \leq k(R) \quad (x, y \in [a, b], \lambda \in (0, 1)) \).
Then the set \( \{ g_\lambda \} := \{ g(\cdot, \lambda), \lambda \in (0,1) \} \) is a subset of real continuous functions defined on \([a,b]\) which satisfies the assumptions of Ascoli-Arzelà’s theorem; hence there exists a sequence \( \lambda_n \) such that:

\[
\lambda_n \to 0
\]

\[
g_{\lambda_n} \to g \text{ for some } g \text{ continuous.}
\]

Observe that, since \( F \) is continuous, from Theorem 2.2 we get the differentiability of \( f \) with respect to \( u \).

Hence for any \( x \in [a,b] \) we have \( g(x) = f'_u(x,u(x)) \).

The rest of the proof consists in showing that the Nemitskii operator \( G \) induced by \( f'_u \) maps \( C^{0,\alpha}(\Omega, \mathbb{R}) \) into itself and is bounded, so that we can apply Theorem 1.1 to prove the claim. For \( u \in C^{0,\alpha}(\Omega, \mathbb{R}) \) with \( \| u \|_\alpha \leq R \) we have \( |g_{\lambda_n}(x)| \leq k(R) \), and thus passing to the limit as \( n \to \infty \), we get \( |g(x)| \leq k(R) \), which implies \( \| G(u) \|_\infty \leq k(R) \). Likewise, letting \( n \to \infty \) in the inequality \( |x-y|^{-\alpha} |g_{\lambda_n}(x) - g_{\lambda_n}(y)| \leq k(R) \), we get \( |x-y|^{-\alpha} |g(x) - g(y)| \leq k(R) \), whence \( h_\alpha(G(u)) \leq k(R) \). We conclude that \( \| G(u) \|_\alpha \leq 2k(R) \) and finish the proof.

**Corollary 3.2.** Let \( \Omega = (a,b) \). Then \( F \) maps \( C^{0,\alpha}(\Omega, \mathbb{R}) \) into itself and is locally lipschitzian if and only if both \( f \) and \( f'_u \) satisfy the assumption (K).

**4. Differentiability.**

**Theorem 4.1.** Let \( f \) be twice differentiable with respect to \( u \) and assume that both \( f \) and \( f'_u \) satisfy the assumption (K), while \( f''_u \) satisfies the assumption (H). Then \( F \) is continuously differentiable.

**Proof:** From the assumptions and Theorem 2.1 the Nemitskii operator \( G \) induced by \( f'_u \) is continuous. Let us compute:

\[
w(x, u, v) = f(x, u(x) + v(x)) - f(x, u(x)) - f'_u(x, u(x))v(x)
\]

\[
= \int_0^1 [f'_u(x, u(x) + \xi v(x)) - f'_u(x, u(x))v(x)] \, d\xi
\]

\[
= \int_0^1 [G(u + \xi v) - G(u)]v(x) \, d\xi
\]

whence

\[
\| F(u + v) - F(u) - G(u)v \|_\alpha \leq \int_0^1 \| G(u + \xi v) - G(u)v \|_\alpha \, d\xi.
\]

Moreover,

\[
|x-y|^{-\alpha} |w(x, u, v) - w(y, u, v)| \leq \int_0^1 |x-y|^{-\alpha} |(G(u + \xi v) - G(u))(x)v(x) - (G(u + \xi v) - G(u))(y)v(y)| \, d\xi
\]

whence

\[
h_\alpha[F(u + v) - F(u) - G(u)v] \leq \int_0^1 h_\alpha[G(u + \xi v) - G(u)v] \, d\xi.
\]
We conclude that
\[ \|F(u + v) - F(u) - G(u)v\|_\alpha \leq \int_0^1 \|(G(u + \xi v) - G(u))v\|_\alpha d\xi \]
\[ \leq m\|v\|_\alpha \int_0^1 \|G(u + \xi v) - G(u)\|_\alpha d\xi. \]

Now let \( \varepsilon > 0 \). By the continuity of \( G \) there exists \( \delta > 0 \) such that \( \|G(u + \xi v) - G(u)\|_\alpha < \varepsilon \) whenever \( \|v\|_\alpha < \delta \). Therefore,
\[ \|F(u + v) - F(u) - G(u)v\|_\alpha \leq \varepsilon \|v\|_\alpha \]
whenever \( \|v\|_\alpha < \delta \), showing that \( F \) is differentiable at \( u \) with derivative \( F'(u)[v] = G(u)v \). Finally, to show that the derivative is continuous, let \( L \) denote the Banach space of all linear bounded mappings of \( C^{0,\alpha}(\overline{\Omega}, \mathbb{R}) \) into itself, equipped with its usual norm \( \|T\|_L = \sup\{\|T[v]\|_\alpha : \|v\|_\alpha = 1\} \). Since
\[ \|F'(u + w)[v] - F'(u)[v]\|_\alpha = \|G(u + w)v - G(u)v\|_\alpha \leq m\|G(u + w) - G(u)\|_\alpha \|v\|_\alpha \]
we have
\[ \|F'(u + w) - F'(u)\|_L \leq m\|G(u + w) - G(u)\|_\alpha \]
and the conclusion follows again from the continuity of \( G \). \( \square \)

Remark. If \( \Omega \) denotes, as before, an open bounded subset of \( \mathbb{R}^n \), the conditions stated in Sections 1, 2, 3, 4 are sufficient also in the case \( f = f(x, u) = f(x, u_1, \ldots, u_m) \) is a real valued function defined in \( \overline{\Omega} \times \mathbb{R}^m \), \( (m \geq 1) \). In this case \( f'_u \) denotes the gradient of \( f \) with respect to the variable \( u \in \mathbb{R}^m \), while \( f''_u \) will denote the \( m \times m \) Hessian matrix \( (f''_{u_iu_j}) (i, j = 1, \ldots, m) \) of \( f \) with respect to the same variable. As a norm in \( C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m) \) we take \( \|u\|_{\alpha,m} = \sum_{i=1}^m \|u\|_{\alpha,i}, \ (u = (u_1, \ldots, u_m)) \).

References


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