Bifurcation for some semilinear elliptic equations 
when the linearization has no eigenvalues 

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Abstract. We prove existence and bifurcation results for a semilinear eigenvalue problem in $\mathbb{R}^N$ ($N \geq 2$), where the linearization $-\Delta$ has no eigenvalues. In particular, we show that under rather weak assumptions on the coefficients $\lambda = 0$ is a bifurcation point for this problem in $H^1, H^2$ and $L^p$ ($2 \leq p \leq \infty$).

Keywords: bifurcation point, variational method, eigenvalues, exponential decay, standing waves

Classification: 35P30, 35A30

1. Introduction and presentation of the results.

In the present paper, we consider the nonlinear eigenvalue problem

(1.1) \[ -\Delta u - q(x)|u|^\sigma_1 u + r(x)|u|^\sigma_2 u = \lambda u \quad \text{in} \quad \mathbb{R}^N, \]

where $N \geq 2$ and $\sigma_1$ and $\sigma_2$ are positive constants such that $\sigma_1 < 4/N$. In particular, we are interested in the question if $\lambda = 0$ is a bifurcation point for the equation (1.1).

Since the problem (1.1) is considered in $\mathbb{R}^N$, the linearization $-\Delta$ has no eigenvalues and $\lambda = 0$ is the infimum of the spectrum of $-\Delta$. In case that $r \equiv 0$, this problem has been studied by many authors. See for instance [5–7], [9], [13–18] and the literature quoted therein. In case that $r \not\equiv 0$, we only know some existence results for the equation (1.1) (see [1], [2], [8] and [12]), but no bifurcation results. In the following, we will close this gap by presenting some bifurcation results for the general case.

We always assume that the functions $q$ and $r$ satisfy the subsequent conditions:

(A) The functions $q, r : \mathbb{R}^N \to \mathbb{R}$ are measurable and $r$ fulfills $r(x) \geq 0$ for almost all $x \in \mathbb{R}^N$.

(B) There exist a constant $0 < a \leq 2 - (\sigma_1 N/2)$ and an open ball $B \subset \mathbb{R}^N$, satisfying $B \neq \emptyset$ and $0 \notin \overline{B}$ ($\overline{B}$ is the closure of $B$), such that $q(x) \geq f(x)|x|^{-a}$ holds for almost all $x \in \zeta$, where $\zeta = \{tx; t \geq 1, x \in B\}$ and $f : \zeta \to [0, \infty)$ is a measurable function satisfying $f(x) \to \infty$ as $|x| \to \infty$.

Moreover, we assume that there exists a constant $K$ such that

\[ r(x) \leq K|x|^b \quad \text{holds for almost all} \quad x \in \zeta, \]
where $b$ is defined by $b = (2 - a)(\sigma_2/\sigma_1) - 2$.

(C) The functions $r$ and $q_- = \min(q, 0)$ are locally integrable.

(D) The function $q_+ = \max(q, 0)$ can be written as $q_+ = q_1 + q_2$, where

(D1) the function $q_1$ satisfies $0 \leq q_1 \in L^\infty$, and $q_1(x)$ tends uniformly to zero as $|x| \to \infty$,

(D2) and the function $q_2$ satisfies $0 \leq q_2 \in L^{p_0}$ for some constant

$$2N/(4 - \sigma_1 N) < p_0 < \infty.$$ 

We want to point out that the above assumptions allow the function $q$ to decay exponentially to $-\infty$ or faster in some direction, and allow the function $r$ to increase exponentially to $+\infty$ or faster in some direction.

**Theorem 1.1.** Suppose that the functions $q$ and $r$ satisfy the assumptions (A)–(D) and that the constant $a$ is defined as in condition (B). Then, there exists a constant $\mu_a \in (0, \infty]$, depending on $a$, such that for each $\mu \in (0, \mu_a)$ there exists a nonpositive constant $\lambda(\mu)$ and a nontrivial nonnegative function $u_\mu \in H^1 \cap L^\infty$ which solves equation (1.1) in the sense of distributions. In case that $a = 2 - (\sigma_1 N/2)$, we have $\mu_a = \infty$. Moreover, it follows that $\lambda(\mu) \to 0$, $\|u_\mu\|_{H^1} \to 0$ and, if $p \in [2, \infty]$, that $\|u_\mu\|_p \to 0$ as $\mu \to 0$. Hence, $\lambda = 0$ is a bifurcation point for equation (1.1) in $H^1$ and in $L^p$ for $p \in [2, \infty]$.

**Corollary 1.2.** (a) If $q_-, r \in L^p_{\text{loc}}$ holds for some constant $p > N/2$, then $u_\mu$ is positive and locally Hölder continuous.

(b) If $q$ and $r$ are locally Hölder continuous, then we have $u_\mu \in C^2$ and the equation (1.1) holds in the classical sense.

**Corollary 1.3.** Suppose in addition to (A)–(D) that $p_0 \geq 2$ and that $q, r \in L^\infty + L^2$. Then, it follows that $u_\mu \in H^2$ and that $\|u_\mu\|_{H^2} \to 0$ as $\mu \to 0$. Thus, $\lambda = 0$ is a bifurcation point for (1.1) in $H^2$.

**Remark 1.4.** In case that $r \equiv 0$, Corollary 1.3 improves Theorem 2.6 (c) in [13]. In [13] it is assumed that $q$ is nonnegative, that $q = q_+$ satisfies condition (D) and that $p_0 \geq 2$. Moreover, it is assumed

(i) that there exist constants $A > 0$ and $0 \leq t < 2 - (\sigma_1 N/2)$ such that $q(x) \geq A(1 + |x|)^{-t}$ holds a.e. in $\mathbb{R}^N$. In case that $N \geq 3$ the author requires additionally

(ii) that $\sigma_1 < 2/(N - 2)$ and $p_0 > 2N/(2 - \sigma_1(N - 2))$. Hence, Corollary 1.3 shows that the condition (i) can be weakened considerably and that condition (ii) is superfluous.

The solutions of the equation (1.1) supply standing waves for nonlinear Klein-Gordon and Schrödinger equations. So, from the standpoint of physics it is an interesting question if the solutions of (1.1) decay exponentially to 0 at infinity.

For the proof of the exponential decay to 0 we need an additional assumption:

(E) There exists a constant $R_0 > 0$ such that $q_2$ satisfies

$$q_2(x) = 0 \text{ for almost all } |x| \geq R_0.$$
Theorem 1.5. Suppose that \( \sigma_2 \leq \sigma_1 \) and that the functions \( q \) and \( r \) satisfy the assumptions (A)–(E). Then, for each \( \mu \in (0, \mu_0) \) the function \( u_\mu \) decays exponentially to 0 at infinity.

Theorem 1.6. Suppose that \( \sigma_1 < \sigma_2 \) and that the functions \( q \) and \( r \) satisfy the assumptions (A)–(E). Then, there exists a decreasing sequence \( (\mu_n) \subset (0, \mu_0) \) such that \( \lim_{n \to \infty} \mu_n = 0 \) and \( u_\mu \) decays exponentially to 0 at infinity.

The proofs for Theorem 1.5–1.6 can be found in § 4.

2. Some preliminaries.

For \( p \in [1, \infty] \), \( L^p = L^p(\mathbb{R}^N) \) and \( L^p_{\text{loc}} = L^p_{\text{loc}}(\mathbb{R}^N) \) are the usual Lebesgue spaces and \( \| \cdot \|_p \) is the norm on \( L^p \). If \( 1 < p < \infty \), then the dual index \( p' \) of \( p \) is defined by \( p' = p/(p - 1) \). Furthermore, \( H^k \) \((k = 1, 2)\) is the Hilbert space \( H^k(\mathbb{R}^N) = W^{k,2}(\mathbb{R}^N) \). The norm on \( H^1 \) is given by \( \| u \|_{H^1} = (\| \nabla u \|^2 + \| u \|^2)^{1/2} \) and the norm on \( H^2 \) by \( \| u \|_{H^2} = (\| \triangle u \|^2 + \| \nabla u \|^2 + \| u \|^2)^{1/2} \). Finally, \( C_0^\infty = C_0^\infty(\mathbb{R}^N) \) denotes the set of all functions which have compact support and derivatives of any order.

If \( N = 2 \), then it follows from the Sobolev imbedding theorem that for each \( p \in [2, \infty) \) there exists a constant \( A_p \) such that

\[
\| u \|_p \leq A_p \| u \|_{H^1} \quad \text{holds for all } u \in H^1.
\]

In case that \( N \geq 3 \), we define \( 2^* = 2N/(N - 2) \). Then, there exists a constant \( C_0 \) such that

\[
\| u \|_{2^*} \leq C_0 \| \nabla u \|_2 \quad \text{holds for all } u \in H^1.
\]

In particular we see that for each \( p \in [2, 2^*] \) there exists a constant \( B_p \) such that

\[
\| u \|_p \leq B_p \| u \|_{H^1} \quad \text{holds for all } u \in H^1.
\]

Let \( F \) be one of the Banach spaces \( H^1 \), \( H^2 \) or \( L^p \). Then a real number \( \lambda \) is called a bifurcation point for the equation (1.1) in \( F \) if and only if there exists a sequence \( (\lambda_n, u_n) \subset \mathbb{R} \times F \) such that \( u_n \not\equiv 0 \), \( \lambda_n \to \lambda \), \( \| u_n \|_F \to 0 \) \((n \to \infty)\) and

\[
\int \nabla u_n \nabla \varphi \, dx - \int q|u_n|^{\sigma_1}u_n \varphi \, dx + \int r|u_n|^{\sigma_2}u_n \varphi \, dx = \lambda_n \int u_n \varphi \, dx
\]

holds for all \( \varphi \in C_0^\infty \) and \( n \in \mathbb{N} \).

When the domain of integration is not indicated, it is understood to be \( \mathbb{R}^N \).

Lemma 2.1. Let \( v \in H^1 \) be a nonnegative function. Then, there exists a sequence \((\varphi_n)\) of nonnegative functions \( \varphi_n \in C_0^\infty \) such that

\[
\varphi_n \to v \quad \text{in } H^1.
\]

Proof: The functions \( \eta_n \) \((n \in \mathbb{N})\) may be chosen such that \( \eta_n \in C_0^\infty \), \( 0 \leq \eta_n \leq 1 \), \( \eta_n(x) = 1 \) holds for \( |x| \leq n \), \( \eta_n(x) = 0 \) if \( |x| \geq n + 1 \) and \( \| \nabla \eta_n \|_\infty \leq C \), where the constant \( C \) is independent of \( n \). Then \( \eta_n v \to v \) in \( H^1 \).

For a function \( u \in L^1_{\text{loc}} \), the regularization \( u_\varepsilon \) may be defined as in [3, p. 147]. Then, we can find a sequence \((\varepsilon_n)\) of positive numbers \( \varepsilon_n \), satisfying \( \varepsilon_n \to 0 \), such that \( \varphi_n = (\eta_n v)_{\varepsilon_n} \to v \) in \( H^1 \).
Lemma 2.2. Let $v \in H^1$ be a nonnegative function and, for $t > 0$, $v_t$ may be defined by $v_t = \min(v, t)$. Then it follows that $v_t \in H^1$, $\partial_i v_t = \partial_i v$ holds almost everywhere in $\{x; v(x) \leq t\}$ and $\partial_i v_t = 0$ holds almost everywhere in $\{x; v(x) > t\}$. Moreover, for each $s \in [1, \infty)$, we have $0 \leq v_t^s \in H^1 \cap L^\infty$ and $\partial_i v_t^s = s v_t^{s-1} \partial_i v_t$ ($i = 1, \ldots, N$).

Proof: The first part of the lemma follows from Lemma 1.1 in [10] and Theorem 7.8 in [3]. The functions $\eta_n$ and the regularizations $u_\varepsilon$ may be defined as in the proof of Lemma 2.1. Then, there exists a sequence of positive numbers $(\varepsilon_n)$ such that $\varepsilon_n \to 0$ and

$$\varphi_n = (\eta_n v_t)\varepsilon_n \to v_t \quad \text{in} \quad H^1.$$  

Here, the functions $\varphi_n$ satisfy $\varphi_n \in C_0^\infty$ and $0 \leq \varphi_n \leq t$. Since $\varphi_n \to v_t$ in $L^2$, we can find a subsequence $(\varphi_{n(k)})$ of $(\varphi_n)$ such that $\varphi_{n(k)}(x) \to v_t(x)$ for almost all $x \in \mathbb{R}^N$.

Now, suppose that $s > 1$. Then it follows that $\varphi_{n(k)}^s \in C_0^1$ and that

$$\partial_i \varphi_{n(k)}^s = s \varphi_{n(k)}^{s-1} \partial_i \varphi_{n(k)}.$$  

Moreover, since $|v_t^s - \varphi_{n(k)}^s| \leq s|v_t - \varphi_{n(k)}| t^{s-1}$, we see that $\varphi_{n(k)}^s \to v_t^s$ in $L^2$. Hence, we obtain: $\partial_i v_t^s = s v_t^{s-1} \partial_i v_t$. \qed

The following lemma can be found in [11, p. 93].

Lemma 2.3. Suppose that $\varphi(t)$ $(t \in [t_0, \infty))$ is a nonnegative and nonincreasing function such that $\varphi(h) \leq C(h - t)^{-\gamma} \varphi(t)^\delta$ holds for all $h > t \geq t_0$. The constants $\gamma$ and $C$ are assumed to be positive and $\delta$ may satisfy $\delta > 1$. Then, for $d = C^{1/\gamma} \varphi(t_0)^{(\delta-1)/\gamma} 2^\delta/(\delta-1)$ it follows that $\varphi(t_0 + d) = 0$.

3. Proof of the main results.

In the present paragraph, we will prove Theorem 1.1 and Corollary 1.2–1.3. We start with

Lemma 3.1. There exist positive constants $\alpha$ and $\beta$, and for each $\varepsilon > 0$ a constant $K_\varepsilon > 0$, such that

$$(2 + \sigma_1)^{-1} \int q_+ |u|^{2+\sigma_1} \, dx \leq \varepsilon \|\nabla u\|_2^2 + K_\varepsilon \left(\|u\|_2^{2+\alpha} + \|u\|_2^{2+\beta}\right)$$

holds for all $u \in H^1$.

Proof: For $\varepsilon = \frac{1}{4}$, the proof can be found in [5, pp. 568–569]. For general $\varepsilon > 0$, the proof proceeds quite similarly. \qed

The nonlinear functional $\xi$ may be defined by

$$\xi(u) = \frac{1}{2} \int |\nabla u|^2 \, dx - (2 + \sigma_1)^{-1} \int q |u|^{2+\sigma_2} \, dx$$

$$+ (2 + \sigma_2)^{-1} \int r |u|^{2+\sigma_2} \, dx.$$
By $D$, we denote the set

$$D = \{u \in H^1; \int |q_-||u|^{2+\sigma_1} \, dx < \infty \text{ and } \int r|u|^{2+\sigma_2} \, dx < \infty\}.$$ 

Moreover, for $\mu \geq 0$, we define $D_\mu = \{u \in D; \|u\|_2 \leq \mu\}$. Then, according to Lemma 3.1, we see that $I(\mu) = \inf_{u \in D_\mu} \xi(u)$ is a well defined real number.

**Lemma 3.2.** (a) Suppose that the constant $a$ in condition (B) satisfies $a = 2 - (\sigma_1 N/2)$. Then it follows that $I(\mu) < 0$ holds for all $\mu > 0$.

(b) Suppose that $a < 2 - (\sigma_1 N/2)$. Then, there exists a constant $\mu_a > 0$ such that $I(\mu) < 0$ holds for all $\mu \in (0, \mu_a)$.

**Remark 3.3.** In the following, we define $\mu_a = \infty$ if $a = 2 - (\sigma_1 N/2)$.

**Proof of Lemma 3.2:** The ball $B$ may be defined as in condition (B) and $\nu$ may be a positive constant. Then, the function $\varphi_0 \in C_0^\infty$ may be chosen such that $\text{supp } \varphi_0 \subset B$ and $\|\varphi_0\|_2 = \nu$. Moreover, for each $t \geq 1$, we define $\varphi_t(x) = t^k \varphi_0(t^{-1}x)$, where $k = (a - 2)/\sigma_1$. Since $\|\varphi_t\|_2 = \nu t^{k+(N/2)}$, we see that $\varphi_t \in D_{\nu t^{k+(N/2)}}$ and that

$$I\left(\nu t^{k+(N/2)}\right) \leq \xi(\varphi_t) = t^{2k+N-2}\left(\frac{1}{2} \int |\nabla \varphi_0(x)|^2 \, dx \right)$$

$$- t^{2+2\sigma_1} (2+\sigma_1)^{-1} \int_B q(tx)|\varphi_0(x)|^{2+\sigma_1} \, dx$$

$$+ t^{2+2\sigma_2} (2+\sigma_2)^{-1} \int_B r(tx)|\varphi_0(x)|^{2+\sigma_2} \, dx$$

$$\leq t^{2k+N-2}\left(\frac{1}{2} \int |\nabla \varphi_0(x)|^2 \, dx \right)$$

$$- \inf_{x \in B} f(tx)(2+\sigma_1)^{-1} \int_B |x|^{-a}|\varphi_0(x)|^{2+\sigma_1} \, dx$$

$$+ \mathcal{K}(2+\sigma_2)^{-1} \int_B |x|^b|\varphi_0(x)|^{2+\sigma_2} \, dx.$$ 

Since $\inf_{x \in B} f(tx) \to \infty$ as $t \to \infty$, we can find a constant $t_0 \geq 1$ such that

$$I\left(\nu t^{k+(N/2)}\right) < 0 \text{ holds for all } t > t_0.$$ (3.1)

Now, suppose that $a = 2 - (\sigma_1 N/2)$. Then, we have $k + (N/2) = 0$. Hence, the part (a) of the lemma follows from (3.1) for $\nu = \mu$. In case that $a < 2 - (\sigma_1 N/2)$, we have $k + (N/2) < 0$. Then, the assertion of the part (b) follows from (3.1) if we define $\nu = 1$, $\mu_a = t_0^{k+(N/2)}$ and $\mu = t^{k+(N/2)}$. \qed
Lemma 3.4. For each $\mu \in (0, \mu_0)$ there exists a function $u_\mu \in D_\mu$ such that $u_\mu \geq 0$, $\|u_\mu\|_2 > 0$ and $\xi(u_\mu) = I(\mu)$.

Proof: Let $\mu \in (0, \mu_0)$, and $(v_n) \subset D$ may be a sequence such that $\xi(v_n) \to I(\mu)$. Then, we may assume without restriction that $\xi(v_n) \leq 0$ and that $v_n \geq 0$ holds for all $n$. Hence, we obtain from Lemma 3.1:

$$
\frac{1}{4} \|\nabla v_n\|_2^2 + (2 + \sigma_1)^{-1} \int |q_-|v_n|^{2+\sigma_1} \, dx
$$

(3.2)

$$
+ (2 + \sigma_2)^{-1} \int r|v_n|^{2+\sigma_1} \, dx \leq K_{1/4}(\mu^{2+\alpha} + \mu^{2+\beta}).
$$

Since $(v_n)$ is bounded in $H^1$, we can find a subsequence of $(v_n)$, still denoted by $(v_n)$, and a $u_\mu \in H^1$ such that $v_n \rightharpoonup u_\mu$ in $H^1$ and $v_n(x) \to u_\mu(x)$ for almost all $x \in \mathbb{R}^N$. Then, it follows from the uniform boundedness principle, (3.2) and Fatou’s lemma that $\|u_\mu\|_2 \leq \mu$, $\|\nabla u_\mu\|_2 \leq \liminf \|\nabla u_n\|_2$,

$$
\int |q_-|u_\mu|^{2+\sigma_1} \, dx \leq \liminf \int |q_-|v_n|^{2+\sigma_1} \, dx < \infty
$$

and

$$
\int r|u_\mu|^{2+\sigma_2} \, dx \leq \liminf \int r|v_n|^{2+\sigma_2} \, dx < \infty.
$$

Moreover, we see that $u_\mu \geq 0$. Since the imbedding $H^1(G) \to L^{2+\sigma_1}p_0(G)$ is compact for all bounded balls $G$ and $q_1(x) \to 0$ as $|x| \to \infty$, it follows that

$$
\int q_+|v_n|^{2+\sigma_1} \, dx \to \int q_+|u_\mu|^{2+\sigma_1} \, dx \quad \text{(see [5, p. 570])}.
$$

Moreover, we obtain

$$
I(\mu) \leq \xi(u_\mu) \leq \liminf \xi(v_n) = I(\mu) < 0
$$

and consequently that $\xi(u_\mu) = I(\mu)$ and $\|u_\mu\|_2 > 0$. \hfill \qed

Lemma 3.5. For $\mu \in (0, \mu_0)$, the function $u_\mu$ may be chosen as in Lemma 3.4. Then, it follows that

$$
\int \nabla u_\mu \nabla \varphi \, dx - \int q|u_\mu|^{\sigma_1}u_\mu\varphi \, dx + \int r|u_\mu|^{\sigma_2}u_\mu\varphi \, dx = \lambda(\mu) \int u_\mu \varphi \, dx
$$

holds for all functions $\varphi \in C_0^\infty$, where

$$
\lambda(\mu) = \|u_\mu\|_2^{-2} \left( \frac{1}{4} \|\nabla u_\mu\|_2^2 - \int q|u_\mu|^{2+\sigma_1} \, dx + \int r|u_\mu|^{2+\sigma_2} \, dx \right).
$$

Proof: Let $\varphi \in C_0^\infty$. Then $d\xi(\|u_\mu\|_2^2|u_\mu + \varepsilon \varphi|^2)^{-1}(u_\mu + \varepsilon \varphi))/d\varepsilon \mid_{\varepsilon = 0} = 0$ implies the assertion. \hfill \qed
Lemma 3.6. The constant $\lambda(\mu)$ may be defined as in Lemma 3.5. Then, we have $\lambda(\mu) \leq 0$.

Proof: For all $t \in (0, 1]$, we have

$$\xi(u_\mu) = I(\mu) \leq I(t\mu) \leq \xi(tu_\mu).$$

Hence $\lambda(\mu) = \|u_\mu\|^2 \frac{2}{d\xi(tu_\mu)/dt} |_{t=1} \leq 0$ implies the assertion.

Proposition 3.7. The constants $\alpha$ and $\beta$ may be chosen as in Lemma 3.1. Then, there exists a constant $C$ such that

$$|\lambda(\mu)| \leq C(\mu^{\alpha} + \mu^{\beta}) \quad \text{and} \quad \|\nabla u_\mu\|^2 \leq C(\mu^{2+\alpha} + \mu^{2+\beta})$$

holds for all $\mu \in (0, \mu_a)$. Hence, $\lambda = 0$ is a bifurcation point for the equation (1.1) in $H^1$.

Proof: Since $\xi(u_\mu) < 0$, we obtain from Lemma 3.1 that

$$\|\nabla u_\mu\|^2 \leq 4K_{1/4}(\|u_\mu\|^2 + \|u_\mu\|^2) \leq 4K_{1/4}(\mu^{2+\alpha} + \mu^{2+\beta}).$$

Moreover, since $\lambda(\mu) \leq 0$, it follows from (3.3) and Lemma 3.1 that

$$|\lambda(\mu)| = -\lambda(\mu) \leq \|u_\mu\|^2 \frac{2}{\int q_+|u_\mu|^2 \sigma_1 dx} \leq (2 + \sigma_1)(4K_{1/4} + K_1)(\|u_\mu\|^2 + \|u_\mu\|^2) \leq C(\mu^{\alpha} + \mu^{\beta}).$$

Lemma 3.8. For all nonnegative functions $v \in H^1$ we obtain

$$\int \nabla u_\mu \nabla v \, dx \leq \lambda(\mu) \int u_\mu v \, dx + \int q_+u_\mu^{1+\sigma_1} v \, dx$$

and, according to Lemma 3.6, that

$$\int \nabla u_\mu \nabla v \, dx \leq \int q_+u_\mu^{1+\sigma_1} v \, dx.$$ 

Proof: Clearly, the assertion holds for all nonnegative functions $v \in C_0^\infty$. Hence, the result follows from Lemma 2.1.

Lemma 3.9. Suppose that $N \geq 3$ and that $\int q_+u_\mu^{1+\sigma_1+s} \, dx < \infty$ holds for some constant $s > 1$. Then, it follows that $u_\mu \in L^{2^*(s+1)/2}$.

Proof: For $t > 0$, the function $v_t$ may be defined by $v_t = \min(u_\mu, t)$. Then, according to Lemma 2.2, we see that $0 \leq v_t^s \in H^1$. Inserting $v_t^s$ in (3.5) shows that

$$4s(s+1)^{-2} \int \nabla v_t^{(s+1)/2} \, dx \leq \int q_+u_\mu^{1+\sigma_1+s} \, dx.$$ 

Hence, using (2.2) and letting $t \to \infty$, we obtain the assertion by Fatou’s lemma.
\textbf{Lemma 3.10.} For each \( p \in [2, \infty) \), we have \( u_\mu \in L^p \).

\textbf{Proof:} For \( N = 2 \) and for \( p \in [2, 2^*] \), if \( N \geq 3 \), the assertion follows from the Sobolev imbedding theorem. Now, suppose that \( N \geq 3 \) and that the constants \( r_n \) and \( s_n \) are defined by \( r_n = 2^*(1 + \varepsilon_0)^n \) and \( s_n = (r_n/p_0') - 1 - \sigma_1 \), where \( \varepsilon_0 = (2^*/2p_0') - (\sigma_1/2) - 1 \). Here, the constant \( p_0 \) is defined as in condition (D2). Since \( p_0 > 2N/(4 - \sigma_1 N + 2\sigma_1) \) and \( r_n \geq 2^* \), it follows that \( \varepsilon_0 > 0 \) and \( s_n > 1 \).

Now, assume that \( u_\mu \in L^{r_n} \) holds for some \( n \in \mathbb{N}_0 \). Then \( 2 \leq 1 + \sigma_1 + s_n < (1 + \sigma_1 + s_n)p_0' = r_n \) implies that

\[
\int q_+ u_\mu^{1+\sigma_1+s_n} \, dx < \infty.
\]

Hence, we obtain from Lemma 3.9 that \( u_\mu \in L^{2^*(s_n+1)/2} \). But

\[
(2^*/2)(s_n + 1) = (2^*/2)((r_n/p_0') - \sigma_1) \\
\geq (2^*/2)(r_n/p_0') - (r_n/2)\sigma_1 \\
= r_n(1 + \varepsilon_0) = r_{n+1}
\]

implies that \( u_\mu \in L^{r_{n+1}} \). Hence, we see that \( u_\mu \in L^p \) holds for all \( p \in [2^*, \infty) \). \( \Box \)

\textbf{Lemma 3.11.} For each \( \mu \in (0, \mu_a) \), we have \( u_\mu \in L^\infty \).

\textbf{Proof:} For \( t > 0 \), we define the function \( U_t \) by \( U_t = (u_\mu - t)_+ \) and the set \( A(t) \) by \( A(t) = \{ x \; ; \; u_\mu(x) \geq t \} \). Then, we obtain from (3.5) that

\[
(3.6) \quad \int \nabla u_\mu \nabla U_t \, dx \leq \int_{A(t)} q_+ u_\mu^{2+\sigma_1} \, dx.
\]

The constant \( p_1 \) may be defined by \( p_1 = 2N/(4 - \sigma_1 N) \). Since \( p_0 > p_1 \), we can find a constant \( p_2 \in (1, \infty) \) such that \( 1/p_0', 1/p_2' = 1/p_1' \). Then, the inequality (3.6) implies

\[
(3.7) \quad \int |\nabla U_t|^2 \, dx \leq C(\mu)(\text{meas } A(t))^{1/p_1'}
\]

for all \( t > 0 \), where \( C(\mu) \) is defined by

\[
(3.8) \quad C(\mu) = \| q_1 \|_\infty \left( \int u_\mu^{(2+\sigma_1)p_1} \, dx \right)^{1/p_1} \\
+ \| q_2 \|_{p_0} \left( \int u_\mu^{(2+\sigma_1)p_0'p_2} \, dx \right)^{1/(p_0'p_2)}.
\]

Now, let us assume that \( N \geq 3 \). Then, it follows from (2.2) and (3.7) that

\[
(3.9) \quad \left( \int_{A(t)} (u_\mu - t)^{2^*} \, dx \right)^{2/2^*} \leq C_0^2 C(\mu)(\text{meas } A(t))^{1/p_1'}.
\]
Moreover, for each $h > t$, we have

$$\left( \int_{A(t)} (u_\mu - t)^2 \, dx \right)^{2/2^*} \geq \left( \int_{A(h)} (u_\mu - t)^2 \, dx \right)^{2/2^*} \geq (h - t)^2 (\text{meas } A(h))^{2/2^*}. \quad (3.10)$$

Combining (3.9) and (3.10) yields

$$\text{meas } A(h) \leq C_0^2 C(\mu)^{2^*/2} (h - t)^{-2^*} (\text{meas } A(t))^{2^*/2p'_1}$$

for all $h > t > 0$. Since $2^*/(2p'_1) = 1 + (\sigma_1 N)/2(N - 2) > 1$, it follows from Lemma 2.3 that $u_\mu$ is essentially bounded. Moreover, for each $t_0 > 0$, we have

$$\|u_\mu\|_{\infty} \leq d + t_0.$$  

where $d = C_0 C(\mu)^{1/2} (\text{meas } A(t_0))^{\sigma_1/4} 2^{1+(2(N-2)/\sigma_1 N)}$. For $t_0 = \|u_\mu\|_2$, it follows that

$$\text{meas } A(t_0) \leq \|u_\mu\|_2^{-2} \int_{A(t_0)} u_\mu^2 \, dx \leq 1.$$  

Hence, we obtain that

$$\|u_\mu\|_{\infty} \leq C_0 C(\mu)^{1/2} 2^{1+(2(N-2)/\sigma_1 N)} + \mu. \quad (3.11)$$

Finally, we consider the case that $N = 2$. Here, we obtain for all $t > 0$:

$$\int U_t^2 \, dx \leq \int_{A(t)} u_\mu^2 \, dx \leq \left( \int_{A(t)} u_\mu^{2p_1} \, dx \right)^{1/p_1} (\text{meas } A(t))^{1/p'_1}. \quad (3.12)$$

Combining (3.7) and (3.12) yields

$$\|U_t\|^2_{H^1} \leq C^*(\mu) (\text{meas } A(t))^{1/p'_1}$$

for all $t > 0$, where

$$C^*(\mu) = C(\mu) + \left( \int u_\mu^{2p_1} \, dx \right)^{1/p_1}. \quad (3.13)$$

Hence, (2.1) implies

$$\left( \int_{A(t)} (u_\mu - t)^p \, dx \right)^{2/p} \leq C_p^2 C^*(\mu) (\text{meas } A(t))^{1/p'_1}$$

for all $t > 0$ and $p \in [2, \infty)$. Then, proceeding as in the case that $N \geq 3$, one can show that

$$\text{meas } A(h) \leq C_p^p C^*(\mu)^{p/2} (h - t)^{-p} (\text{meas } A(t))^{p/(2p'_1)}$$

holds for all $h > t > 0$ and $p \in [2, \infty)$. Hence, according to Lemma 2.3, we see that $u$ is essentially bounded and that

$$\|u_\mu\|_{\infty} \leq C_p C^*(\mu)^{1/2} 2^{(p/(2p'_1))((p/2p'_1) - 1)} + \mu \quad (3.14)$$

if $p > 2p'_1$.
Lemma 3.12. For all \( p \in [2, \infty) \) we have \( \|u_\mu\|_p \to 0 \) as \( \mu \to 0 \).

Proof: We start with the case that \( N = 2 \). Then, according to (2.1), we obtain:

\[
\|u_\mu\|_p \leq C_p \|u_\mu\|_{H^1} \quad \text{for all} \quad \mu \in (0, \mu_a).
\]

Hence, the assertion follows from Proposition 3.7. In case that \( N \geq 3 \) and \( p \in [2, 2^*] \), the assertion is obtained by (2.3) and Proposition 3.7. Now, assume that \( N \geq 3 \) and that \( p \in (2^*, \infty) \). Then, we can find a constant \( t > 0 \) such that \( p = (1 + (t/2))2^* \). Thus, by the Sobolev inequality (2.2), we see that

\[
\|u_\mu\|_{2^* + t}^2 = \|u_\mu^{1+(t/2)}\|_{2^*}^2 \leq C_0^2 \|\nabla u_\mu^{1+(t/2)}\|_2^2
\]

(3.15)

\[
= C_0^2 (1 + (t/2))^{2(1 + t)^{-1}} \int \nabla u_\mu \nabla u_\mu^{1+t} \, dx.
\]

The right hand side of (3.15) is well defined since \( u_\mu \) is bounded. From (3.5), we conclude that

\[
\int \nabla u_\mu \nabla u_\mu^{1+t} \, dx \leq \int q u_\mu^{2 + \sigma_1 + t} \, dx
\]

(3.16)

\[
\leq \|q_1\|_{\infty} \int u_\mu^{2 + \sigma_1 + t} \, dx + \|q_2\|_{p_0} \left( \int \left( u_\mu^{(2 + \sigma_1 + t)p_0'} \, dx \right)^{1/p_0'} \right).
\]

Since

\[
p_0' < 2N/(2(N - 2) + \sigma_1 N) < 2N/(2(N - 2) + \sigma_1 (N - 2))
\]

\[
\leq (2N + tN)/((2 + \sigma_1)(N - 2) + t(N - 2))
\]

\[
= (2 + \sigma_1 + t)^{-1} \cdot (2N + tN)/(N - 2)
\]

\[
= (2 + \sigma_1 + t)^{-1} p,
\]

we see that there is a constant \( \tau \in (0, 1) \) such that

\[
(2 + \sigma_1 + t)p_0' = \tau p + (1 - \tau)2.
\]

Hence, by Hölder’s inequality, we obtain

\[
\left( \int u_\mu^{(2 + \sigma_1 + t)p_0'} \, dx \right)^{1/p_0'} \leq \|u_\mu\|_{p_0'}^{p_\tau/p_0'} \|u_\mu\|_2^{2(1-\tau)/p_0'}.
\]

Then, using again the fact that \( p_0' < 2N/(2(N - 2) + \sigma_1 N) \), it is not difficult to show that \( p_\tau/p_0' < 2 + t \).

Quite similarly, one can prove that there exist constants \( c_1 \in (0, 2 + t) \) and \( c_2 > 0 \) such that \( \int u_\mu^{2 + \sigma_1 + t} \, dx \leq \|u_\mu\|_{p_0'}^{c_1} \|u_\mu\|_2^{c_2} \). Hence, we conclude from (3.15), (3.16) and Young’s inequality that \( \|u_\mu\|_p \to 0 \) as \( \mu \to 0 \). \( \square \)
Lemma 3.13. We have \( \|u_\mu\|_\infty \to 0 \) as \( \mu \to 0 \).

Proof: The constants \( C(\mu) \) and \( C^*(\mu) \) may be defined as in (3.8) and (3.13). Then, according to Lemma 3.12, it follows that \( C(\mu) \to 0 \) and \( C^*(\mu) \to 0 \) as \( \mu \to 0 \). Hence, the assertion follows from (3.11) and (3.14).

Proof of Corollary 1.2: Suppose that the assumptions of part (a) are fulfilled. Then, according to Lemma 3.5, we see that
\[
- \triangle u_\mu + c(x)u_\mu = 0 \quad \text{holds in } \mathcal{D}'(\mathbb{R}^N),
\]
where \( c(x) = -q(x)u^{\sigma_1}_\mu(x) + r(x)u^{\sigma_2}_\mu(x) - \lambda(\mu) \). Since \( p_0 > N/2 \) and \( u_\mu \in L^\infty \), we see that \( c \in L^{p_1}_{\text{loc}} \), where \( p_1 = \min(p_0, p) \) satisfies \( p_1 > N/2 \). Now, the assertion follows from Theorem 7.1 and Corollary 8.1 in [10].

Next, we suppose that the assumptions of the part (b) are fulfilled. Then, it follows from part (a) that \( u_\mu \) is locally Hölder continuous. Hence, the distribution \( \triangle u_\mu \) can be represented by a locally Hölder continuous function. Thus, the assertion of the part (b) follows by a well known result from the regularity theory of elliptic differential equations.

Proof of Corollary 1.3: According to Lemma 3.5, we see that
\[
(3.17) \quad - \triangle u_\mu = \lambda(\mu)u_\mu + qu^{1+\sigma_1}_\mu - ru^{1+\sigma_2}_\mu \quad \text{holds in } \mathcal{D}'(\mathbb{R}^N).
\]
Then, it follows from the assumptions and from Lemma 3.10 – Lemma 3.13 that the right hand side of (3.17) defines a function \( F_\mu \in L^2 \) such that \( \|F_\mu\|_2 \to 0 \) as \( \mu \to 0 \). Consequently, we see that \( u_\mu \in H^2 \) and that \( \|u_\mu\|_{H^2} \to 0 \) as \( \mu \to 0 \).

4. Exponential decay.

Lemma 4.1. Suppose that the functions \( q \) and \( r \) satisfy the assumptions (A)–(E) and that for \( \mu \in (0, \mu_a) \) the function \( u_\mu \) and the constant \( \lambda(\mu) \) are defined as in Lemma 3.4 resp. Lemma 3.5. Moreover, we assume that \( \lambda(\mu) < 0 \) holds for some \( \mu \in (0, \mu_a) \). Then, for each \( c \in (0, -\lambda(\mu)) \) there exists a constant \( A_c \) such that
\[
u_\mu(x) \leq A_c \exp(-(\lambda(\mu) - c)^{1/2}|x|)
\]
holds for almost all \( x \in \mathbb{R}^N \).

Proof: Using the fact that \( u_\mu \) is bounded, we conclude from (D1) and (E) that there exists a constant \( R_c > R_0 \) such that
\[
q_+(x)u^{\sigma_1}_\mu(x) \leq c \quad \text{holds for almost all } x \in \{y; \ |y| > R_c\}.
\]
The function \( \psi \) may be defined by
\[
\psi(x) = A_c \exp(-(\lambda(\mu) - c)^{1/2}|x|) \quad (x \in \mathbb{R}^N).
\]
Here, the constant $A_c$ may be chosen such that
\begin{equation}
\psi(x) \geq u_\mu(x) \quad \text{holds for almost all } x \in \{y; \ |y| \leq R_c\}.
\end{equation}
Then it follows that $\psi \in H^1$ and that
\begin{equation}
\int \nabla \psi \nabla v \, dx \geq (\lambda(\mu) + c) \int \psi v \, dx
\end{equation}
holds for all nonnegative functions $v \in H^1$.

Inequality (4.2) shows that $(u_\mu - \psi)_+$ is a nonnegative function on $H^1$ satisfying $(u_\mu - \psi)_+(x) = 0$ for almost all $x \in \{y; \ |y| \leq R_c\}$. Hence, we obtain from (3.4), (4.1) and (4.3) that
\begin{equation}
\|\nabla (u_\mu - \psi)_+\|^2 = \int \nabla (u_\mu - \psi) \nabla (u_\mu - \psi)_+ \, dx
\leq \lambda(\mu) \int u_\mu (u_\mu - \psi)_+ \, dx + c \int u_\mu (u_\mu - \psi)_+ \, dx
- (\lambda(\mu) + c) \int \psi (u_\mu - \psi)_+ \, dx
= (\lambda(\mu) + c) \|(u_\mu - \psi)_+\|^2_2 \leq 0
\end{equation}
and consequently that $u_\mu \leq \psi$. \hfill \box

**Lemma 4.2.** Let $q$ and $r$ satisfy the assumptions (A)–(D) and suppose that $\sigma_2 \leq \sigma_1$. Then $\lambda(\mu) < 0$ holds for all $\mu \in (0, \mu_a)$.

**Proof:** Since $\xi(u_\mu) < 0$, we see that
\begin{equation}
\int r|u_\mu|^{2+\sigma_2} \, dx < -((2 + \sigma_2)/2)\|\nabla u_\mu\|^2_2 + ((2 + \sigma_2)/(2 + \sigma_1)) \int q|u_\mu|^{2+\sigma_1} \, dx
\end{equation}
and that
\begin{equation}
\lambda(\mu) < \|u_\mu\|^{-2}_2 \left(-(\sigma_2/2)\|\nabla u_\mu\|^2_2 + ((\sigma_2 - \sigma_1)/(2 + \sigma_1)) \int q|u_\mu|^{2+\sigma_1} \, dx \right).
\end{equation}
Then using the fact that
\begin{equation}
\int q|u_\mu|^{2+\sigma_1} \, dx > - (2 + \sigma_1)\xi(u_\mu) > 0,
\end{equation}
we obtain the assertion. \hfill \box

Now, we consider the case that $\sigma_1 < \sigma_2$. Since $I(\cdot)$ is a monotone decreasing function on $[0, \mu_a)$, we can find a measurable subset $\mathcal{M}$ of $[0, \mu_a)$ such that $[0, \mu_a) \setminus \mathcal{M}$ has measure zero and $I(\cdot)$ is differentiable on $\mathcal{M}$ (see [4, Theorem 17.12]). Then, we see that
\begin{equation}
I'(\mu) \leq 0 \quad \text{holds for all } \mu \in \mathcal{M}.
\end{equation}
Lemma 4.3. The function $I(\cdot)$ is Lipschitz continuous on $[0, \mu_a)$ and for all $\mu \in \mathcal{M}$ we have $I'(\mu) \geq \mu^{-1} ||u_\mu||_2^2 \lambda(\mu)$.

Proof: Let $0 \leq \nu < \mu < \mu_a$. Then, we obtain

$$I(\nu) \leq \xi((\nu/\mu)u_\mu)$$

and therefore that

$$I(\nu) - I(\mu) \leq \frac{1}{2}((\nu/\mu)^2 - 1) \int |\nabla u_\mu|^2 \, dx$$

$$- (2 + \sigma_1)^{-1}((\nu/\mu)^{2+\sigma_1} - 1) \int q|u_\mu|^{2+\sigma_1} \, dx$$

$$+ (2 + \sigma_2)^{-1}((\nu/\mu)^{2+\sigma_2} - 1) \int r|u_\mu|^{2+\sigma_2} \, dx.$$  \hspace{1cm} (4.5)

Thus, (4.5) implies for $\mu \in \mathcal{M}$: $I'(\mu) \geq \mu^{-1} ||u_\mu||_2^2 \lambda(\mu)$. Moreover, we obtain

$$|I(\mu) - I(\nu)||\mu - \nu|^{-1} = (I(\nu) - I(\mu))(\mu - \nu)^{-1}$$

$$\leq (2 + \sigma_1)^{-1}(1 - (\nu/\mu)^{2+\sigma_1})(\mu - \nu)^{-1} \int q_+|u_\mu|^{2+\sigma_1} \, dx$$

$$\leq (1 - (\nu/\mu))(\mu - \nu)^{-1} \int q_+|u_\mu|^{2+\sigma_1} \, dx$$

$$= \mu^{-1} \int q_+|u_\mu|^{2+\sigma_1} \, dx.$$

Hence, Lemma 3.1 and Proposition 3.7 show that

$$|I(\mu) - I(\nu)| \leq C(\mu^{1+\alpha} + \mu^{1+\beta})|\mu - \nu|.$$ \hspace{1cm} $\square$

Lemma 4.4. There exists a monotone decreasing sequence $(\mu_n) \subset (0, \mu_a)$ such that $\lim_{n \to \infty} \mu_n = 0$ and $\lambda(\mu_n) < 0$ holds for all $n$.

Proof: Suppose that $\lambda(\mu) \geq 0$ holds for all $\mu \in (0, \mu_a)$. Then, according to Lemma 3.6, we see that $\lambda(\mu) = 0$ holds for all $\mu \in (0, \mu_a)$. Furthermore, (4.4) and Lemma 4.3 would imply that $I'(\mu) = 0$ for all $\mu \in \mathcal{M}$ and consequently that $I(\cdot)$ is constant on $[0, \mu_a)$ (see [4, Theorem 18.15]). In particular, we would obtain that

$$0 = I(0) = I(\min((\mu_a/2), 1)) < 0.$$ 

Hence, there exists a constant $\mu_1 \in (0, \mu_a)$ such that $\lambda(\mu_1) < 0$. Now, repeating this procedure, we can find a $\mu_2 \in (0, \min(\mu_1, 1/2))$ such that $\lambda(\mu_2) < 0$. Moreover, by induction we can show that for each $n$ there is a constant $\mu_n \in (0, \min(\mu_{n-1}, 1/n))$ so that $\lambda(\mu_n) < 0$. \hspace{1cm} $\square$

Finally, we see that Lemma 4.1 and Lemma 4.2 imply Theorem 1.5 and that Theorem 1.6 is obtained by Lemma 4.1 and Lemma 4.4.
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(Received June 12, 1992)