Köthe dual of Banach sequence spaces
\( \ell_p[X] \ (1 \leq p < \infty) \) and Grothendieck space

Wu Congxin, Bu Qingying

Abstract. In this paper, we show the representation of Köthe dual of Banach sequence spaces \( \ell_p[X] \ (1 \leq p < \infty) \) and give a characterization of that the spaces \( \ell_p[X] \ (1 < p < \infty) \) are Grothendieck spaces.

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Let \( X \) be a Banach space and \( X^* \) its topological dual, and let \( B_X \) denote the closed unit ball of \( X \). For \( 1 \leq p < \infty \), let
\[
\ell_p(X) = \left\{ \bar{x} = (x_j) \in X^N : \left\| \bar{x} \right\|_{\ell_p} = \left( \sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p} < \infty \right\},
\]
\[
\ell_p[X] = \left\{ \bar{x} = (x_j) \in X^N : \text{ for each } f \in X^*, \sum_{i \geq 1} |f(x_i)|^p < \infty \right\}.
\]
And for each \( \bar{x} \in \ell_p[X] \), let
\[
\left\| \bar{x} \right\|_{(\ell_p)} = \sup \left\{ \left( \sum_{i \geq 1} |f(x_i)|^p \right)^{1/p} : f \in B_{X^*} \right\}.
\]
Then \( (\ell_p(X), \left\| \cdot \right\|_{\ell_p}) \) and \( (\ell_p[X], \left\| \cdot \right\|_{(\ell_p)}) \) are Banach spaces (see [1], [2], [3]). For \( \bar{x} \in X^N \), let
\[
\bar{x} (i \leq n) = (x_1, \ldots, x_n, 0, 0, \ldots),
\]
\[
\bar{x} (i > n) = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots).
\]
And let
\[
\ell_p[X]_r = \left\{ \bar{x} \in \ell_p[X] : \lim_{n} \| \bar{x} (i > n) \|_{(\ell_p)} = 0 \right\}.
\]
If \( \ell_p[X]_r = \ell_p[X] \), then \( \ell_p[X] \) is said to be a GAK-space [4].

For a vector-valued sequence space \( S(X) \) from \( X \), define its Köthe dual with respect to the dual pair \( (X, X^*) \) (see [4]) as follows:
\[
S(X)^\times |_{(X, X^*)} = \left\{ \bar{f} = (f_j) \in X^{*N} : \text{ for each } \bar{x} = (x_j) \in S(X), \sum_{i \geq 1} |f_i(x_i)| < \infty \right\}.
\]
We denote \( S(X)^\times |_{(X, X^*)} \) by \( S(X)^\times \) simply if the meaning is clear from the context.

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Lemma 1. For $1 \leq p < \infty$, $(\ell_p[X], r)^{\ast} = \ell_p[X]^\ast$.

Proof: It is easy to see that $\ell_p[X]^\ast \subseteq (\ell_p[X], r)^{\ast}$. So we only need to prove that $(\ell_p[X], r)^{\ast} \subseteq \ell_p[X]^\ast$.

For $\overline{x} = (x_j) \in \ell_p[X]$ and $t = (t_j) \in c_0$, let $t\overline{x} = (t_j x_j)$. Then $\|t\overline{x}(i > n)\|_{(\ell_p)} \leq \|\overline{x}\|_{(\ell_p)} \sup_{i > n} |t_i|$ implies that $t\overline{x} \in \ell_p[X]_r$. So for $\overline{f} = (f_j) \in (\ell_p[X], r)^{\ast}$, we have

$$\sum_{i \geq 1} |f_i(t_i x_i)| < \infty.$$ 

It follows from the fact that $t \in c_0$ was taken arbitrary that

$$\sum_{i \geq 1} |f_i(x_i)| < \infty.$$ 

Thus, $\overline{f} \in \ell_p[X]^\ast$ and the proof is completed. \hfill \Box

Lemma 2. (1) For $1 \leq p < \infty$, $\ell_p[X]^\ast \subseteq (\ell_p[X], \|\cdot\|_{(\ell_p)})^\ast$ and $(\ell_p[X], r)^{\ast} = (\ell_p[X], r, \|\cdot\|_{(\ell_p)})^\ast$.

(2) Let $\|\cdot\|_{(\ell_p)}^\ast$ denote the dual norm of $\|\cdot\|_{(\ell_p)}$ on the dual space $(\ell_p[X], \|\cdot\|_{(\ell_p)}^\ast)$. Then for each $\overline{x} \in \ell_p[X]$, we have

$$\|\overline{x}\|_{(\ell_p)}^\ast = \sup \{|\langle \overline{x}, \overline{f} \rangle| : \overline{f} \in \ell_p[X]^\ast, \|\overline{f}\|_{(\ell_p)^\ast} \leq 1\},$$

where $\langle \overline{x}, \overline{f} \rangle = \sum_{i \geq 1} f_i(x_i)$.

Proof: See Theorem 2.3 in [3]. \hfill \Box

Lemma 3. Every weak* unconditionally Cauchy series in $X^\ast$ is weak unconditionally Cauchy series.

Proof: See the proof of p. 49, Corollary 11 in [5]. \hfill \Box

Lemma 4. For $1 \leq p < \infty$,

$$\ell_p[X^\ast] = \left\{ \overline{f} = (f_j) \in X^\ast\mathbb{N} : \text{for each } x \in X, \sum_{i \geq 1} |f_i(x)|^p < \infty \right\}.$$ 

Proof: Let

$$\Delta = \left\{ \overline{f} = (f_j) \in X^\ast\mathbb{N} : \text{for each } x \in X, \sum_{i \geq 1} |f_i(x)|^p < \infty \right\}.$$ 

By definition, we only need to prove that $\Delta \subseteq \ell_p[X^\ast]$.

Let $\overline{f} \in \Delta$ and $t_j \in \ell_q(1/p + 1/q = 1)$. Then $\sum_{i \geq 1} |f_i(t_i x)| < \infty$ for each $x \in X$. So the series $\sum_{j} t_j f_j$ is weak* unconditionally Cauchy in $X^\ast$ and hence, it is weak unconditionally Cauchy by Lemma 3. That is, $\sum_{i \geq 1} |F(t_i f_i)| < \infty$ for each $F \in X^{**}$. Since $(t_j)$ is arbitrary in $\ell_q$, $\sum_{i \geq 1} |F(f_i)|^p < \infty$ and $\overline{f} = (f_j) \in \ell_p[X^\ast]$.

The proof is completed. \hfill \Box
Lemma 5 (the principle of local reflexivity, [6]). Let $X$ be a normed space and $Z^{**}$ a finite dimensional subspace of $X^{**}$. For $\{F_i\}_1^n \subseteq Z^{**}, \{f_i\}_1^n \subseteq X^*$ and $\varepsilon > 0$, there exists a linear map $T : Z^{**} \to X$ such that $\|T\| \leq 1$ and

$$|f_i(TF_i) - F_i(f_i)| < \varepsilon, \quad i = 1, 2, \ldots, n.$$  

Proposition 6. $\ell_p[X^{**}]^\times |(X^{**},X^*) = \ell_p[X]^\times |(X,X^*)$ ($1 \leq p < \infty$).

Proof: It is easy to see that $\ell_p[X] \subseteq \ell_p[X^{**}]$ implies that

$$\ell_p[X^{**}]^\times |(X^{**},X^*) \subseteq \ell_p[X]^\times |(X,X^*) .$$

So we only need to prove that

$$\ell_p[X]^\times |(X,X^*) \subseteq \ell_p[X^{**}]^\times |(X^{**},X^*) .$$

Let $\bar{f} = (f_j) \in \ell_p[X]^\times |(X,X^*)$ and $\bar{F} = (F_j) \in \ell_p[X^{**}]$. For a fixed $n \in \mathbb{N}$, by Lemma 5, there exists a linear map $T_n : \text{span} \{F_i\}_1^n \to X$ such that $\|T_n\| \leq 1$ and

$$|F_i(f_i)| \leq |f_i(T_n F_i)| + 1/n, \quad i = 1, 2, \ldots, n.$$  

Now we prove that $\{(T_n F_1, \ldots, T_n F_n, 0, 0, \ldots)\}_n=1^{\infty}$ is a bounded subset of $\ell_p[X]$. By Theorem 1.5 in [2], we have

$$\|(T_n F_1, \ldots, T_n F_n, 0, 0, \ldots)\|_{(\ell_p)}$$

$$= \sup\left\{\left\|\sum_{i=1}^n s_i T_n F_i\right\| : s = (s_j) \in B_{\ell_q} \right\} (1/p + 1/q) = 1$$

$$\leq \sup\left\{\|T_n\| \left\|\sum_{i=1}^n s_i F_i\right\| : s \in B_{\ell_q} \right\}$$

$$\leq \sup\left\{\left\|\sum_{i=1}^\infty s_i F_i\right\| : s \in B_{\ell_q} \right\} = \|\bar{F}\|_{(\ell_p)} .$$

So $\{(T_n F_1, \ldots, T_n F_n, 0, 0, \ldots)\}_n=1^{\infty}$ is a bounded subset of $\ell_p[X]$ and hence, $\sigma(\ell_p[X], \ell_p[X]^\times |(X,X^*))$-bounded. Thus, we have

$$\sum_{i=1}^n |F_i(f_i)| \leq \sum_{i=1}^n |f_i(T_n F_i)| + 1 \leq \sup_{n \geq 1}\left\{\sum_{i=1}^n |f_i(T_n F_i)| \right\} + 1 .$$

Because $n \in \mathbb{N}$ is arbitrary, it follows that

$$\sum_{i=1}^\infty |F_i(f_i)| < \infty .$$

So we prove that $\bar{f} = (f_j) \in \ell_p[X^{**}]^\times |(X^{**},X^*)$ and this completes the proof. \qed
Proposition 7. $(\ell_p[X]^* |_{(X,X^*)})^* |_{(X^*,X^{**})} = \ell_p[X^{**}]$ $(1 \leq p < \infty)$.

Proof: By Proposition 6, it is easy to see that

$$\ell_p[X^{**}] \subseteq (\ell_p[X^{**}]^* |_{(X^{**},X^*)})^* |_{(X^*,X^{**})} = (\ell_p[X]^* |_{(X,X^*)})^* |_{(X^*,X^{**})}.$$ 

So we only need to prove that

$$(\ell_p[X^{**}]^* |_{(X^{**},X^*)})^* |_{(X^*,X^{**})} \subseteq \ell_p[X^{**}].$$

Let $\overline{F} = (F_j) \in (\ell_p[X^{**}]^* |_{(X^{**},X^*)})^* |_{(X^*,X^{**})}$. Since $f \in X^*$ and $t = (t_j) \in \ell_q$ $(1/p + 1/q = 1)$ implies that $(t_j f) \in \ell_p[X^{**}]^* |_{(X^{**},X^*)}$, $\sum_{i \geq 1} |F_i(t_i f)| < \infty$. Thus, $\sum_{i \geq 1} F_i(f)^p < \infty$ and hence, $\overline{F} \in \ell_p[X^{**}]$ by Lemma 4. The proof is completed. \hfill \Box

Theorem 8. For $1 \leq p < \infty$, $\ell_p \otimes X$, the injective tensor product of $\ell_p$ and $X$, is isometrically isomorphic to the space $((\ell_p[X], \| \cdot \|(\ell_p))$.

Proof: For each $u = \sum_{i=1}^n t^{(i)} \otimes x_i \in \ell_p \otimes X$, $(t^{(i)} \in \ell_p, x_i \in X)$, define $\overline{u} = (\sum_{i=1}^n t^{(i)} x_i, \sum_{i=1}^n t^{(i)} x_i, \ldots)$. Then

$$\|\overline{u}\|_{(\ell_p)} = \sup \left\{ \left| \sum_{k \geq 1} s_k f(\sum_{i=1}^n t^{(i)} x_i) \right| : f \in B_{X^*}, s \in B_{\ell_q} \right\}$$

$$= \sup \left\{ \left| \sum_{i=1}^n f(x_i) (t^{(i)}, s) \right| : f \in B_{X^*}, s \in B_{\ell_q} \right\}$$

$$= \lambda(u) \quad \text{ (see [7, p. 223])} \quad (1/p + 1/q = 1).$$

Let $M = \sup_{1 \leq i \leq n} \|x_i\|$. It follows from the above equality that

$$\|\overline{u} (j > k)\|_{(\ell_p)} = \sup \left\{ \left| \sum_{i=1}^n f(x_i) (t^{(i)}, s \ (j > k)) \right| : f \in B_{X^*}, s \in B_{\ell_q} \right\}$$

$$\leq M \sup \left\{ \sum_{i=1}^n | (t^{(i)}, s \ (j > k)) | : s \in B_{\ell_q} \right\}.$$ 

Since $B_{\ell_q}$ is weak* compact, Theorem 6.11 in [8] implies that

$$\lim_k \|\overline{u} (j > k)\|_{(\ell_p)} = 0.$$ 

So, $\overline{u} \in \ell_p[X]$, and we can define a map $\varphi : \ell_p \otimes X \to \ell_p[X], \varphi(u) = \overline{u}$. It is easy to see that $\varphi$ is a linear isometrically isomorphic map from $\ell_p \otimes X$ to $\ell_p[X]$. Next, we only need to prove that $\varphi$ is surjective.
For $\mathbf{x} = (x_1, \ldots, x_n, 0, 0, \ldots)$, if we let $u = \sum_{i=1}^{n} e_i \otimes x_i$ (where $e_i = (0, \ldots, 0, 1(i), 0, 0, \ldots)$), then $\mathbf{x} = \varphi(u)$. Notice that $\lim_n \varphi(j \leq n) = \varphi$ for each $\mathbf{x} \in \ell_p[X]'$. So $\varphi$ is surjective and the proof is completed.

For two Banach spaces $X$ and $Y$, let $\mathcal{B}(X, Y)$, $I(X, Y)$ and $N(X, Y)$ denote the class of integral bilinear functionals on $X \times Y$, the class of integral operators from $X$ to $Y$ and the class of nuclear operators from $X$ to $Y$ respectively (see p. 232 and p. 170 in [7]).

**Theorem 9.** Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then $\mathbf{f} = (f_j) \in \ell_p[X]^\times$ if and only if there exist an $r = (r_j) \in \ell_1$ a bounded sequence $\{s^{(n)} \}_{n=1}^\infty$ of $\ell_q$ and a bounded sequence $\{h_n \}_{n=1}^\infty$ of $X^*$ such that

$$f_i = \sum_{n \geq 1} r_n s^{(n)}_i h_n, \quad i = 1, 2, \ldots.$$ 

**Proof:** Necessity. Let $\mathbf{f} = (f_j) \in \ell_p[X]^\times$. By Lemma 1 and Lemma 2, $\mathbf{f} \in (\ell_p[X]' \cap \| \cdot \|_{(\ell_p)}^*)$. So Theorem 8 implies that there is an $\psi \in (\ell_p \overset{\psi}{\otimes} X)^*$ corresponding to $\mathbf{f}$. By Definition 6 in [7, p. 232], there is an $\psi \in \mathcal{B}(\ell_p, X)$ corresponding to $\psi^*$. Furthermore, by Corollary 12 in [7, p. 237], there exists a $T_\psi \in I(\ell_p, X^*)$ corresponding to $\psi$. Since Corollary 10 in [7, p. 235] and Theorem 6 in [7, p. 248] guarantee that $I(\ell_p, X^*) = N(\ell_p, X^*)$, there are an $r = (r_j) \in \ell_1$, a bounded sequence $\{s^{(n)} \}_{n=1}^\infty$ of $\ell_q$ and a bounded sequence $\{h_n \}_{n=1}^\infty$ of $X^*$ such that $T_\psi(t) = \sum_{n \geq 1} r_n \langle t, s^{(n)} \rangle h_n$, for $t \in \ell_p$.

Now for each $i \geq 1$ and each $x \in X$, by the above corresponding relations, we have $T_\psi(e_i)(x) = \psi(e_i, x) = \psi^*(e_i \otimes x) = \langle \varphi(e_i \otimes x), \mathbf{f} \rangle = f_i(x)$.

Thus $$f_i = T_\psi(e_i) = \sum_{n \geq 1} r_n s^{(n)}_i h_n, \quad i = 1, 2, \ldots.$$ 

Sufficiency. Let $M = \sup_{n \geq 1} \| s^{(n)} \|_q$ and $N = \sup_{n \geq 1} \| h_n \|$. Then, for each $\mathbf{x} = (x_j) \in \ell_p[X]$, we have

$$\sum_{i \geq 1} |s^{(n)}_i h_n(x_i)| \leq MN \| \mathbf{x} \|_{(\ell_p)}, \quad \text{for } n \geq 1.$$ 

And so

$$\sum_{i \geq 1} |f_i(x_i)| \leq \sum_{n \geq 1} |r_n| \sum_{i \geq 1} |s^{(n)}_i h_n(x_i)| < \infty.$$ 

Therefore, $\mathbf{f} \in \ell_p[X]^\times$ and the proof is completed.
**Theorem 10.** For $1 < p < \infty$, $(\ell_p[X]^\times, \| \cdot \|_{(\ell_p)^*})$ is a GAK-space.

**Proof:** Let $\mathcal{F} = (f_j) \in \ell_p[X]^\times$. Then by Theorem 9, there exist an $r = (r_j) \in \ell_1$, a bounded sequence $\{s^{(n)}\}_1^\infty$ of $\ell_q$ and a bounded sequence $\{h_n\}_1^\infty$ of $X^*$ such that

$$f_i = \sum_{n \geq 1} r_ns^{(n)}_ih_n, \quad i = 1, 2, \ldots.$$ 

Without loss of generality, we can assume that $\|s^{(n)}\|_q \leq 1$ and $\|h_n\| \leq 1$ for $n \geq 1$. Thus, for $\mathbf{x} \in \ell_p[X]$ with $\|\mathbf{x}\|_{(\ell_p)} \leq 1$, we have

$$\sum_{i \geq 1} |s^{(n)}_ih_n(x_i)| \leq \|\mathbf{x}\|_{(\ell_p)} \leq 1 \quad \text{for } n \geq 1.$$ 

So

$$\left\{ \left( \sum_{i \geq 1} |s^{(n)}_ih_n(x_i)| \right)_{n \geq 1} : \|\mathbf{x}\|_{(\ell_p)} \leq 1 \right\} \subseteq B_{\ell_\infty}.$$

Let $\varepsilon > 0$. Then $B_{\ell_\infty}$ is weak* compact implies that there exists an $n_0 \in \mathbb{N}$ such that

$$\sum_{n > n_0} r_n \sum_{i \geq 1} |s^{(n)}_ih_n(x_i)| < \varepsilon/2, \quad \mathbf{x} \in \ell_p[X], \quad \|\mathbf{x}\|_{(\ell_p)} \leq 1.$$ 

Since $B_{\ell_p}$ is weakly compact set and

$$\left\{ (h_n(x_i))_{i \geq 1} : \mathbf{x} \in \ell_p[X], \quad \|\mathbf{x}\|_{(\ell_p)} \leq 1, \quad n \geq 1 \right\} \subseteq B_{\ell_p},$$

there is a $k_0 \in \mathbb{N}$ such that for each $k > k_0$,

$$\sum_{i > k} |s^{(n)}_ih_n(x_i)| < \varepsilon/2\|r\|_1$$

for $\mathbf{x} \in \ell_p[X]$ with $\|\mathbf{x}\|_{(\ell_p)} \leq 1$ and $n = 1, 2, \ldots, n_0$. Thus, for each $\mathbf{x} \in \ell_p[X]$ with $\|\mathbf{x}\|_{(\ell_p)} \leq 1$ and each $k > k_0$, we have

$$\sum_{i > k} |f_i(x_i)| \leq \sum_{n = 1}^{n_0} r_n \sum_{i > k} |s^{(n)}_ih_n(x_i)| + \sum_{n > n_0} r_n \sum_{i > k} |s^{(n)}_ih_n(x_i)|$$

$$\leq \left( \sum_{n = 1}^{\infty} r_n \right) \varepsilon/2\|r\|_1 + \sum_{n > n_0} r_n \sum_{i \geq 1} |s^{(n)}_ih_n(x_i)| < \varepsilon.$$

So for $k > k_0$,

$$\|\mathbf{f} (j > k)\|_{(\ell_p)}^* = \sup\left\{ |\langle \mathbf{f}, \mathbf{x} \rangle (j > k) \rangle : \mathbf{x} \in \ell_p[X], \quad \|\mathbf{x}\|_{(\ell_p)} \leq 1 \right\}$$

$$= \sup\left\{ |\sum_{i > k} f_i(x_i)| : \|\mathbf{x}\|_{(\ell_p)} \leq 1 \right\} < \varepsilon.$$
Therefore, \( \lim_{k} \|f(j > k)\|^{*}_{(\ell_{p})} = 0 \) and \( f \in (\ell_{p}[X], \| \cdot \|^{*}_{(\ell_{p})})_{r} \).

For \( 1 < p < \infty \), by Theorem 10 and [4, Proposition 4.9], we have

\[
\sigma((\ell_{p}[X]^{\infty})_{(X,X^{*})})^{\infty} |(X^{*},X^{**})| = (\ell_{p}[X]^{\infty} |(X,X^{*}), \| \cdot \|^{*}_{(\ell_{p})})*. \tag{*}
\]

Now, if we let \( \| \cdot \|^{*}_{(\ell_{p})} \) denote the dual norm of \( \| \cdot \|^{*}_{(\ell_{p})} \) on the dual space \( (\ell_{p}[X]^{\infty} |(X,X^{*}), \| \cdot \|^{*}_{(\ell_{p})})* \), then by Proposition 7 and Lemma 2, the norm \( \| \cdot \|^{*}_{(\ell_{p})} \) on the space \( \ell_{p}[X^{**}] \) is equal to the norm \( \| \cdot \|^{*}_{(\ell_{p})} \).

Similarly as the proof of Theorem 3.6 in [3], we have the following two propositions.

**Proposition 11.** Let \( f^{(n)}(n) \in \ell_{p}[X]^{\infty} (1 \leq p < \infty) \). Then that

\[
\sigma((\ell_{p}[X]^{\infty} |(X,X^{*}), (\ell_{p}[X]^{\infty} |(X,X^{*}))^{\infty} |(X^{*},X^{**})) - \lim_{n} f^{(n)} = 0
\]

is equivalent to

(a) \( \sigma(X^{*},X^{**}) - \lim_{n} f^{(n)} = 0 \) for \( i \geq 1 \); and

(b) \( \sup_{n \geq 1} \| f^{(n)} \|^{*}_{(\ell_{p})} < \infty \)

if and only if \( (\ell_{p}[X]^{\infty} |(X,X^{*}))^{\infty} |(X^{*},X^{**}), \| \cdot \|^{*}_{(\ell_{p})} \) is a GAK-space.

**Proposition 12.** Let \( f^{(n)} \in (\ell_{p}[X])^{*} (1 \leq p < \infty) \). Then

\[
\sigma((\ell_{p}[X])^{*},\ell_{p}[X]) - \lim_{n} f^{(n)} = 0
\]

if and only if \( \sigma(X^{*},X) - \lim_{n} f^{(n)} = 0 \) for \( i \geq 1 \) and \( \sup_{n \geq 1} \| f^{(n)} \|^{*}_{(\ell_{p})} < \infty \).

We say a Banach space \( X \) to be a Grothendieck space if every weak* null sequence on \( X^{*} \) is weak null sequence (see [7, p. 179]). Leonard [1] has proved that \( \ell_{p}(X) (1 < p < \infty) \) is a Grothendieck space if and only if \( X \) is a Grothendieck space. Now we have

**Theorem 13.** For \( 1 < p < \infty \). The Banach space \( (\ell_{p}[X], \| \cdot \|^{*}_{(\ell_{p})}) \) is a Grothendieck space if and only if

(i) \( X \) is a Grothendieck space; and
(ii) \( (\ell_{p}[X^{**}], \| \cdot \|^{*}_{(\ell_{p})}) \) is a GAK-space.

**Proof:** Sufficiency. By (ii), \( (\ell_{p}[X], \| \cdot \|^{*}_{(\ell_{p})}) \) is a GAK-space, i.e. \( \ell_{p}[X]_{r} = \ell_{p}[X] \).

Let \( f^{(n)} \in (\ell_{p}[X], \| \cdot \|^{*}_{(\ell_{p})})^{*} \) such that

\[
\sigma(\ell_{p}[X]^{*},\ell_{p}[X]) - \lim_{n} f^{(n)} = 0.
\]
By Proposition 12, we have
\[ \sigma(X^*, X) - \lim_{n} f^{(n)}_i = 0, \quad i = 1, 2, \ldots \]
and
\[ \sup_{n \geq 1} \| \overline{f}^{(n)} \|_{(\ell_p)} < \infty. \]

By (i), we have
\[ \sigma(X^*, X^{**}) - \lim_{n} f^{(n)}_i = 0, \quad i = 1, 2, \ldots . \]

By (ii) and Propositions 2, 6, 7, the space \( ((\ell_p[X]^*)^{\times} \mid (X^*, X^{**}), \| \cdot \|_{(\ell_p)}) \) is a GAK-space. So Proposition 11 guarantees that
\[ \sigma(\ell_p[X]^*, (\ell_p[X]^*)^{\times}) - \lim_{n} \overline{f}^{(n)}(n) = 0. \]

It follows from (*) that
\[ \sigma(\ell_p[X]^*, \ell_p[X]^{**}) - \lim_{n} \overline{f}^{(n)}(n) = 0. \]
and completes the sufficiency.

Necessity. To prove (i), let \( f_n \in X^* (n \geq 1) \) such that
\[ \sigma(X^*, X) - \lim_{n} f_n = 0. \]

Let \( \overline{f}^{(n)} = (f_n, 0, 0, \ldots) \) for \( n \geq 1 \). Then \( \overline{f}^{(n)} \in (\ell_p[X]_r)^* \) and
\[ \sigma((\ell_p[X]_r)^*, \ell_p[X]_r) - \lim_{n} \overline{f}^{(n)}(n) = 0. \]
So
\[ \sigma((\ell_p[X]_r)^*, (\ell_p[X]_r)^{**}) - \lim_{n} \overline{f}^{(n)}(n) = 0 \]
and hence, \( \sigma(X^*, X^{**}) - \lim_{n} f_n = 0 \). (i) follows.

For (ii), let \( \overline{f}^{(n)} \in \ell_p[X]^{\times} \mid (X, X^*) \) such that
\[ \sigma(X^*, X^{**}) - \lim_{n} f^{(n)}_i = 0, \quad i = 1, 2, \ldots \]
and
\[ \sup_{n \geq 1} \| \overline{f}^{(n)} \|_{(\ell_p)} < \infty. \]

By Lemmas 1, 2 and Proposition 12, we have
\[ \sigma((\ell_p[X]_r)^*, \ell_p[X]_r) - \lim_{n} \overline{f}^{(n)} = 0. \]
And hence,
\[ \sigma((\ell_p[X]_r)^*, (\ell_p[X]_r)^{**}) - \lim_{n} \overline{f}^{(n)} = 0. \]

It follows from (*) that
\[ \sigma(\ell_p[X]^* \mid (X, X^*), (\ell_p[X]^* \mid (X, X^*))^{\times} \mid (X^*, X^{**})) - \lim_{n} \overline{f}^{(n)}(n) = 0. \]

So Propositions 6, 7, 11 imply that \( (\ell_p[X]^{**}, \| \cdot \|_{(\ell_p)}) \) is a GAK-space and (ii) follows. The proof is completed. \( \square \)
Corollary 14. If $\ell_p[X]_r$ (1 < $p$ < $\infty$) is a Grothendieck space, then $\ell_p[X]$ is a GAK-space.

References


Department of Mathematics, Harbin Institute of Technology, Harbin, 150006 China

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