Ergodic properties of contraction semigroups in $L_p$, $1 < p < \infty$

RYOTARO SATO

Abstract. Let $\{T(t) : t > 0\}$ be a strongly continuous semigroup of linear contractions in $L_p$, $1 < p < \infty$, of a $\sigma$-finite measure space. In this paper we prove that if there corresponds to each $t > 0$ a positive linear contraction $P(t)$ in $L_p$ such that $|T(t)f| \leq P(t)|f|$ for all $f \in L_p$, then there exists a strongly continuous semigroup $\{S(t) : t > 0\}$ of positive linear contractions in $L_p$ such that $|T(t)f| \leq S(t)|f|$ for all $t > 0$ and $f \in L_p$. Using this and Akcoglu's dominated ergodic theorem for positive linear contractions in $L_p$, we also prove multiparameter pointwise ergodic and local ergodic theorems for such semigroups.

Keywords: contraction semigroup, semigroup modulus, majorant, pointwise ergodic theorem, pointwise local ergodic theorem

Classification: 47A35

1. Introduction and the main result

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $L_p = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, denote the usual Banach spaces of real or complex functions on $(X, \Sigma, \mu)$. A linear operator $T : L_p \to L_p$ is called a contraction if $\|T\|_p \leq 1$, $\|T\|_p$ being the operator norm of $T$ in $L_p$, positive if $0 \leq f \in L_p$ implies $Tf \geq 0$, and majorizable if there exists a positive linear operator $P : L_p \to L_p$ such that $|Tf| \leq P|f|$ for all $f \in L_p$. Any such $P$ will be referred to as a majorant of $T$. It is known (cf. [5, §4.1]) that a bounded linear operator $T$ in $L_p$ possesses a majorant $P$ when $p = 1$ or $\infty$. But this is not the case when $1 < p < \infty$. The Hilbert transform serves as an example in $L_p$ for all $1 < p < \infty$ (see Starr [8]). The following proposition is needed later, whose proof is omitted because it is essentially the same as that of Theorem 4.1.1 in [5].

Proposition (cf. [5], Remark, p. 161). Let $T$ be a bounded linear operator in $L_p$, $1 < p < \infty$, and let $P$ be a majorant of $T$. Then there exists a unique positive linear operator $\tau$ in $L_p$, called the linear modulus of $T$, such that

(i) $\|\tau\|_p \leq \|P\|_p$,
(ii) $|Tf| \leq \tau|f|$ for all $f \in L_p$,
(iii) $\tau f = \sup\{|Tg| : g \in L_p, |g| \leq f\}$ for all $f \in L_p^+$.

From now on let us fix $p$ with $1 < p < \infty$. Let $\{T(t) : t > 0\}$ be a strongly continuous semigroup of linear contractions in $L_p$, i.e.
(i) each $T(t)$ is a linear contraction in $L_p$,  
(ii) $T(t)T(s) = T(t+s)$ for all $t, s > 0$,  
(iii) $\lim_{t \to s} ||T(t)f - T(s)f||_p = 0$ for all $f \in L_p$ and $s > 0$.

Since the operators $T(t)$ are not necessarily majorizable, it cannot be expected that the semigroup $\{T(t) : t > 0\}$ is majorizable by a positive semigroup, i.e. there exists a strongly continuous semigroup $\{S(t) : t > 0\}$ of positive linear operators in $L_p$ such that $|T(t)f| \leq S(t)|f|$ for all $t > 0$ and $f \in L_p$. But if each $T(t)$ possesses a majorant $P(t)$ such that $||P(t)||_p \leq 1$, then we can prove the following main result in this paper.

**Theorem 1** (cf. Theorem 1 in [7]). Let $\{T(t) : t > 0\}$ be a strongly continuous semigroup of linear contractions in $L_p$, $1 < p < \infty$. Suppose each $T(t)$ possesses a majorant $P(t)$ such that $||P(t)||_p \leq 1$. Then there exists a strongly continuous semigroup $\{S(t) : t > 0\}$ of positive linear contractions in $L_p$, called the semigroup modulus of $\{T(t) : t > 0\}$, such that

\begin{enumerate}
  \item $|T(t)f| \leq S(t)|f|$ for all $t > 0$ and $f \in L_p$,
  \item $S(t)f = \sup\{\tau(t_1) \cdots \tau(t_n)f : \sum_{i=1}^{n} t_i = t, t_i > 0, n \geq 1\}$ for all $f \in L_p^+$, where $\tau(t)$ denotes the linear modulus of $T(t)$,
  \item $\tau(0) = \text{strong-lim}_{t \to +0} S(t)$, where $\tau(0)$ denotes the linear modulus of $T(0) = \text{strong-lim}_{t \to +0} T(t)$.
\end{enumerate}

**Proof:** For an $f \in L_p^+$ and $t > 0$, define

\[(1) \quad S(t)f = \sup\{\tau(t_1) \cdots \tau(t_n)f : \sum_{i=1}^{n} t_i = t, t_i > 0, n \geq 1\}.
\]

Since $||\tau(t)||_p \leq ||P(t)||_p \leq 1$ and $\tau(t)\tau(s) \geq \tau(t+s) \geq 0$ for all $t, s > 0$, it follows that

\[(2) \quad ||S(t)f||_p \leq ||f||_p
\]

and that

\[(3) \quad S(t)(cf) = cS(t)f \quad \text{and} \quad S(t)(f+g) = S(t)f + S(t)g
\]

for a constant $c > 0$ and $f, g \in L_p^+$. Thus we may regard $S(t)$ as a positive linear contraction in $L_p$. From the definition of $S(t)$ it easily follows that

\[(4) \quad S(t)S(s) = S(t+s) \quad \text{for all} \quad t, s > 0.
\]

Since (i) is clear, to complete the proof it is enough to establish (iii), because (iii) together with the fact that $||S(t)||_p \leq 1$ for all $t > 0$ implies that for every $f \in L_p$ and $s > 0$

\[
\lim_{t \to +0} ||S(s)f - S(s+t)f||_p \leq \lim_{t \to +0} ||S(s-t)||_p ||S(t)f - S(2t)f||_p \\
\leq \lim_{t \to +0} (||S(t)f - \tau(0)f||_p + ||S(2t)f - \tau(0)f||_p) = 0,
\]
Ergodic properties of contraction semigroups in $L^p$, $1 < p < \infty$

and similarly $\lim_{t \to +0} \|S(s)f - S(s-t)f\|_p = 0$; namely, $\{S(t) : t > 0\}$ is strongly continuous at each $s > 0$. For this purpose we first remark that $T(0) = \text{strong-} \lim_{t \to +0} T(t)$ exists. This is due to Lemma 1 in [6], because $L^p$ is a reflexive Banach space and $\|T(t)\|_p \leq 1$ for all $t > 0$.

We next show that the linear modulus $\tau(0)$ of $T(0)$ exists. To do this, define

$$P(0)f = \sup \{ |T(0)g| : g \in L^p, |g| \leq f \} \quad \text{for } f \in L^+_p.$$  

Since $\lim_{t \to +0} \|T(t)g - T(0)g\|_p = 0$, it follows that there exists a sequence $\{t_n\}$ of positive reals with $t_n \downarrow 0$ for which

$$T(0)g = \lim_n T(t_n)g \quad \text{a.e. on } X.$$  

Then

$$|T(0)g| \leq \liminf_n \tau(t_n)|g| \leq \liminf_n \tau(t_n)f \quad \text{a.e. on } X.$$  

Since there are countable functions $g_i \in L^p$, $1 \leq i \leq \infty$, such that $|g_i| \leq f$ and $P(0)f = \sup_i |T(0)g_i|$ a.e. on $X$, we apply the Cantor diagonal argument to infer that there exists a sequence $\{t_n\}$ of positive reals with $t_n \downarrow 0$ for which

$$P(0)f \leq \liminf_n \tau(t_n)f \quad \text{a.e. on } X.$$  

Then, by Fatou’s lemma,

$$\|P(0)f\|_p \leq \liminf_n \|\tau(t_n)f\|_p \leq \|f\|_p \quad (f \in L^+_p).$$  

It also follows from the proof of Theorem 4.1.1 in [5] that if $\{B_1, \ldots, B_m\}$ is a finite measurable partition of $X$, then

$$\sum_{i=1}^m |T(0)(1_{B_i}f)| \leq P(0)f \quad \text{a.e. on } X,$$

where $1_{B_i}$ denotes the indicator function of $B_i$. Thus we see, as in the proof of Theorem 4.1.1 in [5], that the linear modulus $\tau(0)$ of $T(0)$ exists. (Incidentally we note that $\tau(0)f = P(0)f$ for all $f \in L^+_p$.)

To prove (iii), let $f \in L^+_p$ be fixed arbitrarily, and given an $\varepsilon > 0$ choose $g_i \in L^p$, $1 \leq i \leq n$, so that

$$|g_i| \leq f \quad \text{and} \quad \|\tau(0)f - \max_i |T(0)g_i|\|_p < \varepsilon.$$  

Since $T(0) = \text{strong-} \lim_{t \to +0} T(t)$, choose $\delta > 0$ so that

$$0 < t < \delta \quad \text{implies} \quad \|T(0)g_i - T(t)g_i\|_p < \varepsilon/n \quad (1 \leq i \leq n).$$
Then, putting \( h_0 = \max_i |T(0)g_i| \) and \( h_t = \max_i |T(t)g_i| \) for \( t > 0 \), we get
\[
|h_0 - h_t| \leq \max_i |T(0)g_i - T(t)g_i| \leq \sum_{i=1}^n |T(0)g_i - T(t)g_i|,
\]
and hence \( \|h_0 - h_t\|_p \leq \sum_{i=1}^n \|T(0)g_i - T(t)g_i\|_p < \varepsilon \) for \( 0 < t < \delta \). Thus
\[
\|\tau(0)f - \max_i |T(t)g_i|\|_p \leq \|\tau(0)f - h_0\|_p + \|h_0 - h_t\|_p < \varepsilon + \varepsilon = 2\varepsilon \quad \text{for} \quad 0 < t < \delta,
\]
and since \( S(t)f \geq \tau(t)f \geq \max_i |T(t)g_i| \), it follows that
\[
(\tau(0)f - S(t)f)^+ \leq (\tau(0)f - \max_i |T(t)g_i|)^+.
\]
This yields
\[
\| (\tau(0)f - S(t)f)^+ \|_p \leq \| (\tau(0)f - \max_i |T(t)g_i|)^+ \|_p < 2\varepsilon
\]
for \( 0 < t < \delta \). That is,
\[
(8) \quad \lim_{t \to +0} \| (\tau(0)f - S(t)f)^+ \|_p = 0.
\]

On the other hand, since \( T(t)T(0) = T(0)T(t) = T(t) \) implies \( \tau(t)\tau(0) \geq \tau(t) \) and \( \tau(0)\tau(t) \geq \tau(t) \), it follows that
\[
(9) \quad S(t)\tau(0) \geq S(t) \quad \text{and} \quad \tau(0)S(t) \geq S(t) \quad \text{for all} \quad t > 0.
\]
Therefore
\[
(10) \quad (\tau(0)f - S(t)f)^- \leq (\tau(0)f - S(t)\tau(0)f)^- \leq |\tau(0)f - S(t)\tau(0)f|
\]
and
\[
(11) \quad (\tau(0)f - S(t)\tau(0)f)^+ \leq (\tau(0)f - S(t)f)^+.
\]
By (11) and (8),
\[
\lim_{t \to +0} \| (\tau(0)f - S(t)\tau(0)f)^+ \|_p \leq \lim_{t \to +0} \| (\tau(0)f - S(t)f)^+ \|_p = 0.
\]
Thus
\[
(12) \quad \lim_{t \to +0} \| \tau(0)f - (S(t)\tau(0)f \wedge \tau(0)f) \|_p = 0,
\]
whence

$$\lim_{t \to +0} \int (S(t) \tau(0)f \wedge \tau(0)f)^p d\mu = \|\tau(0)f\|_p^p. \quad (13)$$

We now use the relations

$$0 \leq \left[S(t)\tau(0)f - (S(t)\tau(0)f \wedge \tau(0)f)\right]^p$$

$$\leq \left(S(t)\tau(0)f\right)^p - \left(S(t)\tau(0)f \wedge \tau(0)f\right)^p \quad \text{(because } 1 < p < \infty)$$

and

$$\int (S(t)\tau(0)f)^p d\mu \leq \|\tau(0)f\|_p^p \quad \text{(because } \|S(t)\|_p \leq 1)$$

together with (13) to see that

$$\lim_{t \to +0} \|S(t)\tau(0)f - (S(t)\tau(0)f \wedge \tau(0)f)\|_p = 0. \quad (14)$$

Hence by (12), \(\lim_{t \to +0} \|\tau(0)f - S(t)\tau(0)f\|_p = 0\); and (10) gives

$$\lim_{t \to +0} \|(\tau(0)f - S(t)f)^-\|_p \leq \lim_{t \to +0} \|\tau(0)f - S(t)\tau(0)f\|_p = 0. \quad (15)$$

This and (8) imply that \(\lim_{t \to +0} \|\tau(0)f - S(t)f\|_p = 0\) for all \(f \in L_p^+,\) completing the proof. \(\square\)

2. An application

**Theorem 2** (cf. Theorem VIII.7.10 in [3] and Theorem 4.3 in [4]). Let \(\{T_i(t) : t \geq 0\}, \ i = 1, \ldots, d,\) be strongly continuous semigroups of linear contractions in \(L_p, \ 1 < p < \infty.\) Suppose each \(T_i(t)\) possesses a majorant \(P_i(t)\) such that \(\|P_i(t)\|_p \leq 1.\) Then for every \(f \in L_p\) the averages

$$A(u_1, \ldots, u_d)f(x)$$

$$= \frac{1}{u_1 \ldots u_d} \int_0^{u_1} \cdots \int_0^{u_d} T_1(t_1) \cdots T_d(t_d)f(x) \, dt_1 \cdots dt_d \quad (16)$$

converge a.e. to \(T_1(0) \cdots T_d(0)f(x)\) as \(\max_i u_i \to 0,\) and also they converge a.e. to \(E_1 \cdots E_d f(x)\) as \(\min_i u_i \to \infty,\) where \(E_i\) is the operator in \(L_p\) defined by

$$E_i f = \lim_{b \to \infty} \frac{1}{b} \int_0^b T_i(t) f \, dt \quad \text{in } L_p-\text{norm.}$$

**Proof:** We first show that the function

$$f^*(x) = \sup_{u_1, \ldots, u_d > 0} |A(u_1, \ldots, u_d)f(x)| \quad (x \in X) \quad (17)$$
is in $L_p$ and satisfies $\|f^*\|_p \leq (p/(p - 1))^d \|f\|_p$.

For this purpose let $\{S_i(t) : t > 0\}$, $1 \leq i \leq d$, denote the semigroup moduli of the semigroups $\{T_i(t) : t > 0\}$, $1 \leq i \leq d$. Write for $u > 0$ and $1 \leq i \leq d$,

$$A_i(u)f(x) = \frac{1}{u} \int_0^u T_i(t)f(x)\,dt \quad \text{and} \quad B_i(u)|f|(x) = \frac{1}{u} \int_0^u S_i(t)|f|(x)\,dt.$$  

Since

$$|A_i(u)f(x)| \leq B_i(u)|f|(x) \quad \text{a.e. on } X$$

and

$$\sup_{u>0} B_i(u)|f|(x) = \sup_{u \in Q^+} B_i(u)|f|(x),$$

where $Q^+$ denotes the set of positive rationals, and for every $u \in Q^+$

$$B_i(u)|f| = \lim_{n \to \infty} \frac{1}{u(n)!} \sum_{m=0}^{u(n)!-1} S_i(m/n!)|f| \quad \text{in } L_p\text{-norm},$$

it follows from the Cantor diagonal argument that there exists a subsequence $\{n'\}$ of the sequence of positive integers such that

$$\sup_{u>0} B_i(u)|f|(x) \leq \liminf_{n' \to \infty} f^*_{i,n'}(x) \quad \text{a.e. on } X,$$

where

$$f^*_{i,n}(x) = \sup_{k \geq 1} \frac{1}{k} \sum_{m=0}^{k-1} S_i(m/n!)|f|(x) \quad (n \geq 1).$$

Thus, by Fatou’s lemma and Akcoglu’s dominated ergodic theorem [1] for positive linear contractions in $L_p$ with $1 < p < \infty$,

$$\|\sup_{u>0} B_i(u)|f|(x)\|_p \leq \liminf_{n' \to \infty} \|f^*_{i,n'}\|_p \leq \frac{p}{p - 1} \|f\|_p. \tag{18}$$

Now, the equality $A(u_1, \ldots, u_d)f = A_1(u_1) \ldots A_d(u_d)f$ implies

$$f^*(x) = \sup_{u_1, \ldots, u_d > 0} |A_1(u_1) \ldots A_d(u_d)f(x)|$$

$$\leq \sup_{u_1, \ldots, u_d > 0} B_1(u_1) \ldots B_d(u_d)|f|(x)$$

$$\leq \sup_{u_1, \ldots, u_{d-1} > 0} B_1(u_1) \ldots B_{d-1}(u_{d-1}) (\sup_{u>0} B_d(u)|f|)(x),$$

and hence by induction

$$\|f^*\|_p \leq \left(\frac{p}{p - 1}\right)^{d-1} \|\sup_{u>0} B_d(u)|f|\|_p \leq \left(\frac{p}{p - 1}\right)^d \|f\|_p. \tag{19}$$
We apply (19) to infer that the averages $A(u_1, \ldots, u_d) f(x)$ converge a.e. to $T_1(0) \ldots T_d(0) f(x)$ [resp. $E_1 \ldots E_d f(x)$] as $\max_i u_i \to 0$ [resp. $\min_i u_i \to \infty$], as follows.

We use an induction argument. Since the set
\[
M = \left\{ \frac{1}{b} \int_0^b T_1(t) g(x) \, dt + h : b > 0, \ T_1(0) g = g, \ T_1(0) h = 0 \right\}
\]
is dense in $L^p$, there exists a sequence $\{f_n\}$ in $M$ such that $\lim_n \|f_n - f\|_p = 0$.
Since $f_n \in M$ implies
\[
\lim_{u \to +0} A_1(u) f_n(x) = T_1(0) f_n(x) \quad \text{a.e. on } X,
\]
it follows that the function
\[
F(x) = \limsup_{u \to +0} |A_1(u) f(x) - T_1(0) f(x)| \quad (x \in X)
\]
satisfies
\[
F(x) \leq \limsup_{u \to +0} |A_1(u) (f - f_n)(x) - T_1(0) (f - f_n)(x)|
\]
\[
\leq \sup_{u > 0} |A_1(u) (f - f_n)(x)| + |T_1(0) (f - f_n)(x)|.
\]
Thus
\[
\|F\|_p \leq \frac{p}{p-1} \|f - f_n\|_p + \|f - f_n\|_p \to 0 \quad (n \to \infty).
\]
We get $F(x) = 0$ a.e. on $X$ and hence $\lim_{u \to +0} A_1(u) f(x) = T_1(0) f(x)$ a.e. on $X$.

Next, since $L_p$ is a reflexive Banach space, we see by Eberlein’s mean ergodic theorem (cf. [5, Theorem 2.1.5, p. 76]) that there exists a projection operator $E_1 : L_p \to L_p$ for which
\[
E_1 f = \lim_{u \to \infty} A_1(u) f \quad \text{in } L_p\text{-norm},
\]
and that the set
\[
M^\sim = \{ g + (h - T_1(s) h) : s > 0, \ T_1(t) g = g \quad \text{for all } t > 0 \}
\]
is dense in $L_p$. If $g + (h - T_1(s) h) \in M^\sim$, where $T_1(t) g = g$ for all $t > 0$, then
\[
A_1(u) [g + (h - T_1(s) h)](x)
\]
\[
= g(x) + \frac{1}{u} \int_0^s T_1(t) h(x) \, dt - \frac{1}{u} \int_u^{u+s} T_1(t) h(x) \, dt,
\]
and
\[ \lim_{u \to \infty} \frac{1}{u} \int_0^s T_1(t)h(x) \, dt = 0 \quad \text{a.e. on } X. \]

Letting \( n = [u] \) be the integral part of \( u \) and \( k \) be an integer such that \( s < k - 1 \), we have

\[
\left| \frac{1}{u} \int_u^{u+s} T_1(t)h(x) \, dt \right| \leq \frac{1}{u} \int_u^{u+s} S_1(t)|h(x)| \, dt \\
\leq \frac{1}{n} \int_n^{n+k} S_1(t)|h(x)| \, dt = \frac{1}{n} S_1(n)h^\sim(x),
\]

where
\[
h^\sim(x) = \int_0^k S_1(t)|h(x)| \, dt \quad (x \in X).
\]

Define the functions

\[
(21) \quad H_n(x) = \sum_{m=n}^\infty \left( \frac{1}{m} S_1(m)h^\sim(x) \right)^p \quad (x \in X).
\]

Clearly we get \( H_n \geq H_{n+1} \geq \cdots \geq 0 \) and

\[
\int H_n \, d\mu = \sum_{m=n}^\infty m^{-p} \| S_1(m)h^\sim \|^p_p \leq \left( \sum_{m=n}^\infty m^{-p} \right) \| h^\sim \|^p_p \to 0 \quad (n \to \infty).
\]

It follows that \( \lim_{n} H_n(x) = 0 \) a.e. on \( X \), and

\[
\lim_{u \to \infty} \left| \frac{1}{u} \int_u^{u+s} T_1(t)h(x) \, dt \right| \leq \lim_{n \to \infty} \frac{1}{n} S_1(n)h^\sim(x) = 0
\]

a.e. on \( X \). This proves that

\[
\lim_{u \to \infty} A_1(u)[g + (h - T_1(s)h)](x) = g(x) = E_1[g + (h - T_1(s)h)](x)
\]

a.e. on \( X \). Using this and the density of \( M^\sim \) in \( L_p \), it follows as before that the function

\[
(22) \quad F^\sim(x) = \limsup_{u \to \infty} |A_1(u)f(x) - E_1f(x)| \quad (x \in X)
\]

satisfies \( F^\sim = 0 \) a.e. on \( X \). Thus \( \lim_{u \to \infty} A_1(u)f(x) = E_1f(x) \) a.e. on \( X \).

We then use the relation

\[
A(u_1, \ldots, u_d)f = A(u_1, \ldots, u_{d-1})A_d(u_d)f
\]
to complete the proof. Since the functions

\begin{equation}
    f\sim(u; x) = \sup_{0<r\leq u} |A_d(r)f(x) - T_d(0)f(x)| \quad (x \in X)
\end{equation}

satisfy

\[ 0 \leq f\sim(v; x) \leq f\sim(u; x) \in L_p \quad \text{for} \quad 0 < v < u \]

and

\[ \lim_{u \to +0} f\sim(u; x) = 0 \quad \text{a.e. on} \quad X, \]

and since

\[ A(u_1, \ldots, u_d)f - T_1(0) \ldots T_d(0)f = A(u_1, \ldots, u_{d-1})[A_d(u_d)f - T_d(0)f] + [A(u_1, \ldots, u_{d-1}) - T_1(0) \ldots T_{d-1}(0)](T_d(0)f), \]

it follows from the induction hypothesis that the function

\begin{equation}
    G(x) = \limsup_{u_1 \vee \cdots \vee u_d \to 0} |A(u_1, \ldots, u_d)f(x) - T_1(0) \ldots T_d(0)f(x)| \quad (x \in X)
\end{equation}

satisfies

\[ G(x) \leq \limsup_{u_1 \vee \cdots \vee u_{d-1} \vee u_d \to 0} |A(u_1, \ldots, u_{d-1})[A_d(u_d)f - T_d(0)f](x)| \]

\[ \leq \sup_{u_1, \ldots, u_{d-1} > 0} B_1(u_1) \ldots B_{d-1}(u_{d-1})f\sim(u_d; \cdot)(x) \]

a.e. on X. Hence we get \( \|G\|_p \leq \left( \frac{p}{p-1} \right)^{d-1} \|f\sim(u_d; \cdot)\|_p \to 0 \) as \( u_d \to +0, \) by the Lebesgue dominated converge theorem. This implies that \( A(u_1, \ldots, u_d)f(x) \to T_1(0) \ldots T_d(0)f(x) \) a.e. on X as \( \max_i u_i \to 0. \)

Essentially the same proof can be applied to infer that \( A(u_1, \ldots, u_d)f(x) \to E_1 \ldots E_d f(x) \) a.e. on X as \( \min_i u_i \to \infty, \) and hence we omit the details. \( \square \)

3. Concluding remarks

(a) In Theorem 1 the hypothesis that \( \{T(t) : t > 0\} \) is a contraction semigroup cannot be omitted. In fact, given an \( \varepsilon > 0 \) there exists a strongly continuous semigroup \( \{T(t) : t > 0\} \) of bounded linear operators in \( L_p, 1 < p < \infty, \) such that each \( T(t) \) possesses a majorant \( P(t) \) satisfying \( \|P(t)\|_p < 1 + \varepsilon \) and also such that

\[ \lim_{m \to \infty} \|(\tau(1/m))^m\|_p = \infty, \]

where \( \tau(1/m) \) denotes the linear modulus of \( T(1/m), \ m \geq 1. \) An example can be found in [7].
(b) In Theorem 2 the hypothesis that each $T_i(t)$ possesses a majorant $P_i(t)$ such that $\|P_i(t)\|_p \leq 1$ cannot be omitted. In fact, there are negative examples for $p = 2$. More precisely, Akcoglu and Krengel [2] constructed a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of unitary operators in $L_2$ with $T(0) = \text{identity}$ such that the averages $\frac{1}{u} \int_0^u T(t)f(x)\,dt$ diverge a.e. as $u \to +0$ for some $f$ in $L_2$. Essentially the same idea can be applied to construct another strongly continuous semigroup $\{T(t) : t \geq 0\}$ of unitary operators in $L_2$ with $T(0) = \text{identity}$ such that the averages $\frac{1}{u} \int_0^u T(t)f(x)\,dt$ diverge a.e. as $u \to \infty$ for some $f$ in $L_2$. See also [5, pp. 191–192].

References


Department of Mathematics, Faculty of Science, Okayama University, Okayama, 700 Japan

(Received September 20, 1993)