A primrose path from Krull to Zorn

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Abstract. Given a set $X$ of “indeterminates” and a field $F$, an ideal in the polynomial ring $R = F[X]$ is called conservative if it contains with any polynomial all of its monomials. The map $S \mapsto RS$ yields an isomorphism between the power set $\mathcal{P}(X)$ and the complete lattice of all conservative prime ideals of $R$. Moreover, the members of any system $\mathcal{I} \subseteq \mathcal{P}(X)$ of finite character are in one-to-one correspondence with the conservative prime ideals contained in $P_\mathcal{I} = \bigcup\{RS : S \in \mathcal{I}\}$, and the maximal members of $\mathcal{I}$ correspond to the maximal ideals contained in $P_\mathcal{I}$. This establishes, in a straightforward way, a “local version” of the known fact that the Axiom of Choice is equivalent to the existence of maximal ideals in non-trivial (unique factorization) rings.

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In 1979, Hodges [3] derived a certain maximal principle on trees, equivalent to Zorn’s Lemma and hence to the Axiom of Choice (AC), from the statement that every nontrivial unique factorization domain contains a maximal ideal. In fact, he showed more, namely that if suffices to take into account certain “pseudo-localizations” of polynomial rings (in an arbitrary number of indeterminates) over the rational number field $\mathbb{Q}$. Recently, Banaschewski [1] gave a short and direct deduction of AC from the above specific maximal ideal theorem. Since one argument in his proof involved the infinity of $\mathbb{Q}$, he asked whether an alternative argument might provide the same conclusion over an arbitrary (possibly finite) base field $F$. We shall show that this is in fact the case, by establishing an elementary one-to-one correspondence between the subsets of a fixed set $X$ and so-called conservative prime ideals of the polynomial ring $R = F[X]$. Concerning basic ring-theoretical background, see, for example, the monograph “Commutative rings” by Kaplansky [4].

By a prime set in an arbitrary ring, we mean a proper subset $P$ such that $ab \in P$ if and only if $a \in P$ or $b \in P$. Hence one version of Krull’s Prime Ideal Theorem states that every ideal contained in a prime set $P$ is contained in a prime ideal $Q \subseteq P$. The equivalence of this statement, even for non-commutative rings, with the lattice-theoretical Prime Ideal Theorem (PIT), alias Boolean Ultrafilter Theorem, has been established in [2]. (Notice, however, that in the non-commutative case, a prime ideal need not be a prime set.) By the work of Halpern and Levy, PIT is weaker than AC in BNG set theory (cf. [6, p. 99]).
Henceforth, we focus on the following specific setting: given a set $X$ and an arbitrary field $F$, we are considering the (commutative) polynomial ring $R = F[X]$ with the elements of $X$ as indeterminates. The multiplicative submonoid generated by these indeterminates is the free abelian monoid over $X$. It consists of all (unitary) monomials and will be denoted by $M$. Recall that any polynomial $a \in R$ has a unique representation $q_1m_1 + \cdots + q_nm_n$ as a linear combination of monomials $m_1, \ldots, m_n$ with non-zero coefficients $q_1, \ldots, q_n \in F$. The collection of these $a$-monomials is denoted by $M_a$. For any subset $A$ of $R$, we put $M_A = \bigcup\{M_a; a \in A\}$ and call $A$ ($M$-)conservative if $M_A \subseteq A$. Writing $RS$ for the ideal generated by a subset $S$ of $R$, one immediately observes that an ideal $I$ is conservative iff it is of the form $RS$ for some $S \subseteq M$ (in fact, $I = RM_I$).

The conservative ideals of $R$ form a closure system $\mathcal{C}(R)$, hence a complete lattice. The corresponding closure operator assigns to each $A \subseteq R$ the ideal $RM_A$. The lattice $\mathcal{C}(R)$ is easily seen to be superalgebraic, that is, algebraic and completely distributive: indeed, each conservative ideal $I$ is a join of completely join-prime (= supercompact) members of $\mathcal{C}(R)$, namely of the principal ideals generated by monomials in $I$. Furthermore, not only the join of conservative ideals is conservative, but also the product of any two conservative ideals. In other words, $\mathcal{C}(R)$ is a subquantale of the quantale $\mathcal{I}(R)$ of all ideals (see, for example, [5]). Moreover, the map $S \mapsto RS$ yields an isomorphism between the Alexandrov topology of all ideals of the monoid $M$ (i.e. of all subsets $S$ of $M$ with $mS \subseteq S$ for all $m \in M$) and $\mathcal{C}(R)$. The inverse isomorphism is given by $I \mapsto M_I = M \cap I$. Next, we characterize the ideals of the form $RS$ where $S$ is a set of indeterminates.

**Lemma 1.** The assignment $S \mapsto RS$ yields an isomorphism between the power set $\mathcal{P}(X)$ and the complete lattice of all conservative prime ideals.

**Proof:** It is easily verified that each set $RS$ with $S \subseteq X$ is a conservative prime ideal. Conversely, let $P$ be any conservative prime ideal of $R$. Then, for $a \in P$, each $a$-monomial $m$ belongs to $P$, and as $P$ is prime, $m = rx$ for some $r \in R$ and $x \in S = X \cap P$. Hence the element $a$ is a member of the ideal $RS$, being a linear combination of its monomials. This proves the inclusion $P \subseteq RS$, and the converse inclusion is clear since $P$ is an ideal containing $S$. The equation

$$S = X \cap RS \quad (S \subseteq X)$$

shows that the map $S \mapsto RS$ is one-to-one, with inverse $P \mapsto X \cap P$. Of course, these two mutually inverse maps preserve inclusion and are therefore isomorphisms.

By a primrose of $R$, we mean a subset $P$ of $R$ such that for each $a \in P$, there is some $S \subseteq X$ with $a \in RS \subseteq P$. In view of Lemma 1, the primroses are just the unions of conservative prime ideals, in other words, sets of the form

$$P_{\mathcal{S}} = \bigcup\{RS : S \in \mathcal{S}\}$$
with \( I \subseteq \mathcal{P}(X) \). Clearly, any such union is still a conservative prime set, but the converse does not hold. For example, if \( x \) and \( y \) are distinct indeterminates from \( X \) then the union \( P = Rx \cup Ry \cup R(x + y) \) is a conservative prime set but not a primrose since there is no \( S \subseteq X \) such that \( x + y \in RS \subseteq P \).

Recall that a collection \( \mathcal{I} \) of subsets of \( X \) is a system of finite character (on \( X \)) provided a set \( S \) belongs to \( \mathcal{I} \) if and only if \( E \in \mathcal{I} \) for all finite subsets \( E \) of \( S \). Among the various maximal principles equivalent to the Axiom of Choice (cf. [6]), the most convenient version is here the lemma of Tukey-Teichmüller, stating that any member of a system of finite character is contained in a maximal one.

**Lemma 2.** There is a one-to-one correspondence \( \mathcal{I} \mapsto P_{\mathcal{I}} \) between the systems of finite character on \( X \) and the primroses of \( R \). Moreover, for fixed \( \mathcal{I} \), the map \( S \mapsto RS \) induces a bijection between \( \mathcal{I} \) and the set of all conservative prime ideals contained in \( P_{\mathcal{I}} \).

**Proof:** Given any primrose \( P \), it is straightforward to show that the system

\[
\mathcal{I}_P = \{ S \subseteq X : RS \subseteq P \}
\]

is of finite character, and \( P = P_{\mathcal{I}_P} \).

Clearly, if \( \mathcal{I} \subseteq \mathcal{P}(X) \) is any system of finite character with \( P = P_{\mathcal{I}} \) then we have \( \mathcal{I} \subseteq \mathcal{I}_P \). On the other hand, if \( S \) is a member of \( \mathcal{I}_P \) then for each finite subset \( E = \{ x_1, \ldots, x_n \} \) of \( S \), the element \( x_1 + \cdots + x_n \) belongs to \( RS \subseteq P = P_{\mathcal{I}} \), hence to \( RS' \) for some \( S' \in \mathcal{I} \), so that by Lemma 1, \( E \subseteq S' \). Thus \( E \in \mathcal{I} \) for each finite \( E \subseteq S \), and so \( S \in \mathcal{I} \). This proves the equation \( \mathcal{I} = \mathcal{I}_P \) and shows that the map \( P \mapsto \mathcal{I}_P \) is inverse to the map \( \mathcal{I} \mapsto P_{\mathcal{I}} \).

We now come to a key result.

**Lemma 3.** For any primrose \( P \) and any ideal \( I \subseteq P \), the smallest conservative ideal containing \( I \) is still a subset of \( P \).

**Proof:** First, we prove the inclusion \( Rm + I \subseteq P \) for \( a \in I \) and any \( a \)-monomial \( m \). Let \( b \in I \) and choose an exponent \( n \) large enough such that no \( b \)-monomial has \( m^n \) as a factor. Then \( c = m^n a + b \in I \subseteq P \), hence \( c \in RS \subseteq P \) for some \( S \subseteq X \). As \( m^{n+1} \) and all \( b \)-monomials are \( c \)-monomials, too, one obtains \( m^{n+1} \in RS \) and \( M_b \subseteq RS \). But \( RS \) is a prime ideal by Lemma 1, so that \( Rm + b \subseteq RS \subseteq P \).

Now it is easy to show that the conservative ideal \( RM_I \) is a subset of \( P \): for any finite subset \( E \) of \( M_I \), a straightforward induction gives \( RE + I \subseteq P \), and then it follows that \( RM_I \subseteq P \).

**Corollary.** Any ideal maximal among the ideals contained in a fixed primrose \( P \) is a conservative prime ideal.

For any prime set \( P \subseteq R \), the quotients \( \frac{R}{u} \) with \( r \in R \) and \( u \in R \setminus P \) form a subring \( R_P \) of the quotient field of \( R \), and the canonical embedding of \( R \) in \( R_P \) gives rise to a one-to-one correspondence between the prime ideals of \( R \) contained in \( P \) and the prime ideals of \( R_P \) (cf. [5, 1–5]). We shall refer to \( R_P \) as a pseudo-localization of \( R \). In all, we have established the following
Proposition. Let $X$ be a set, $F$ an arbitrary field, and $R$ the polynomial ring $F[X]$. Then the maximal members of any system $\mathscr{S}$ of finite character on $X$ are in one-to-one correspondence with the maximal ideals contained in $P_{\mathscr{S}}$, and consequently, with the maximal ideals of the pseudo-localization $R_{P_{\mathscr{S}}}$.

This immediately leads to a “local version” of Hodges’ result that the existence of maximal ideals in unique factorization rings of the above type implies the Axiom of Choice.

Corollary. The following two statements on a set $X$ and a polynomial ring $R = F[X]$ are equivalent:

(a) Each system of finite character on $X$ has a maximal member.
(b) Each pseudo-localization $R_P$ by a primrose $P$ has a maximal ideal.

Notice that (a) entails the existence of a set of representatives for any partition $\mathcal{A}$ of $X$, since any such set is a maximal member of the following system of finite character:

$$\mathcal{I} = \{ S \subseteq X : |S \cap A| \leq 1 \text{ for each } A \in \mathcal{A} \}.$$

Corollary. Under the assumption of PIT, for any ideal $I$ contained in a primrose $P$, there is a conservative prime ideal $R_S$ with $I \subseteq R_S \subseteq P$.

Proof: The set of all conservative ideals contained in $P$ is closed under directed unions, and its complement is multiplicatively closed in $\mathcal{C}(R)$. Hence, by the Separation Lemma for Quantales which is equivalent to PIT (see [2]), any conservative ideal $I \subseteq P$ is contained in a conservative prime ideal $R_S \subseteq P$, and Lemma 3 completes the proof. □

Added in proof. It can be shown that PIT is not only sufficient but also necessary for the above conclusion.

References


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