A compact ccc non-separable space from a Hausdorff gap and Martin’s Axiom

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Abstract. We answer a question of I. Juhasz by showing that MA + ¬ CH does not imply that every compact ccc space of countable π-character is separable. The space constructed has the additional property that it does not map continuously onto $I^{\omega_1}$.

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1. Introduction

I. Juhasz [Ju71] has proven that MA($\omega_1$) implies that every first countable compact ccc space is separable. This has been extended by Shapirovskii [Sh72] by replacing first countable with countable tightness. In Juhasz [Ju77], the question is raised whether tightness can be replaced by π-character, i.e., whether MA($\omega_1$) implies that every compact ccc space of countable π-character is separable. We will show not. We present our space as a space whose points are certain ideals because this is the way that we found it; although the inclined reader should easily be able to identify the base set as a rather simple subset of $2^{\omega} \times \kappa$ (where $\kappa$ is a certain regular cardinal $> \omega_1$) using a Hausdorff gap as a parameter.

2. General theory of total ideal spaces

Let $P = \bigcup_{A \subseteq \omega} 2^A$ and put $p \preceq q$ if $q$ extends $p$. Then $(P, \preceq)$ is a Dedekind complete partially ordered set. A subset $F$ of $P$ is compatible if $\bigcup F \in P$. We write $p \parallel q$ if $\{p, q\}$ is compatible and we write $p \perp q$ if $\{p, q\}$ is not compatible. A subset $Q$ of $P$ is closed in $P$ if whenever $F$ is a finite compatible subset of $Q$, then $\bigcup F \in Q$. For $Q$ closed in $P$, a compatible and closed subset $I$ of $Q$ is called a total ideal of $Q$ if

(a) $\bigcup I$ has domain all of $\omega$ and
(b) $p \in I$ and $q \in Q$ with $q \preceq p$ implies $q \in I$.

Let Fin = $\{p \in P : \text{dom}(p) \text{ is finite}\}$. The parameter for these ideal spaces will be a closed subset $Q$ of $P$ with Fin $\subseteq Q$. For such a $Q$, put $X(Q) = \{I \subseteq Q : I$ is a total ideal of $Q\}$. It is seen that $X(Q)$ is a closed subspace of $2^Q$ (where

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2 = \{0, 1\} has the discrete topology, $2^Q$ has the product topology, and points of $X(Q)$ are identified with their characteristic functions. For $B \subset Q$, put $B^+ = \{I \in X(Q) : B \subset I\}$ and $B^- = \{I \in X(Q) : B \cap I = \emptyset\}$. If $B = \{q\}$, then we simply write $q^+$ and $q^-$. Then, since $Q$ is closed, a base for $X(Q)$ consists of the clopen sets $q^+ \cap B^-$ where $q \in Q$ and $B$ is a finite subset of $\{r \in Q : q < r\}$.

We note the helpful facts that

\begin{align*}
(\mathrm{a}) & \quad q^+ \subset r^+ \iff r \preceq q, \\
(\mathrm{b}) & \quad q^+ \cap r^+ \neq \emptyset \iff q \parallel r \iff [q^{-1}(1) \cup r^{-1}(1)] \cap [q^{-1}(0) \cup r^{-1}(0)] = \emptyset, \\
(\mathrm{c}) & \quad \lambda : X(Q) \to 2^\omega \text{ given by } \lambda(I) = \bigcup I \text{ is a continuous surjection.}
\end{align*}

Put $Q = \{q^+ : q \in Q\}$ and for each $f \in 2^\omega$, let $M_f$ be the maximal total ideal $\{q \in Q : q \preceq f\}$.

**Fact 2.1.** $Q$ is a $T_0$-separating, binary $\pi$-base of $X(Q)$. Hence, $\pi w(X(Q)) = c f(Q, \preceq) = \min\{\langle D \rangle : D \subset Q \text{ and } \forall q \in Q : \exists d \in D : (q \preceq d)\}$.

This was proved in [Be88] and [Be89]. $T_0$-separating and binary are straightforward. The fact that $Q$ is a $\pi$-base is crucial. The reader will see a proof of this in Lemma 3.2 where we must prove a little bit more in order to achieve countable $\pi$-character.

Now we show two facts which delineate the kinds of Souslinian examples that we can get from these ideal spaces.

**Fact 2.2.** If $X(Q)$ is $\sigma$-linked, then $X(Q)$ is separable.

If $X(Q)$ is $\sigma$-linked, then $Q = \bigcup_{n<\omega} Q_n$ where for each $n < \omega$, $Q_n$ is linked. Since $Q$ is binary, by choosing $I_n \in \bigcap Q_n$ for each $n < \omega$, we get that $\{I_n : n < \omega\}$ is dense in $X(Q)$.

We refer the reader to Todorcevic [To89] for the definition of the Open Colouring Axiom OCA.

**Fact 2.3** (OCA). If $X(Q)$ is ccc, then $X(Q)$ is separable.

For each $q \in Q$ put $A_q = q^{-1}(1)$ and $B_q = q^{-1}(0)$. Then $A_q$ and $B_q$ are disjoint subsets of $\omega$. For each $q \in Q$ let $a_q$, $b_q$ be the characteristic functions of $A_q$, $B_q$ respectively. Let $S = \{(a_q, b_q) : q \in Q\}$ have the subspace topology from $2^\omega \times 2^\omega$. Define a partition of $[S]^2$ by $\{(a_q, b_q), (a_r, b_r)\} \in K_0$ iff $q^+ \cap r^+ = \emptyset$ iff $(A_q \cup A_r) \cap (B_q \cup B_r) \neq \emptyset$. $K_0$ is open in $[S]^2$. Since $X(Q)$ is ccc, there does not exist a $K_0$-homogeneous subset of $S$ which has cardinality $\omega_1$. Hence, by OCA, $Q = \bigcup_{n<\omega} Q_n$ where for every $n$ and for every $q, r$ in $Q_n$, $q^+ \cap r^+ \neq \emptyset$, i.e., $\{q^+ : q \in Q_n\}$ is linked. We get that $Q$ is $\sigma$-linked, hence Fact 2.2 implies that $X(Q)$ is separable.

**Remarks:** We have learned that Fact 2.3 follows from a more general result Theorem 10.3* in Todorcevic and Farah [TF95]. We see from Fact 2.3, that if we want a ccc but not separable space $X(Q)$, then we must be in a model of set theory contradicting OCA. We did this in [Be89] under CH producing a first
countable Corson compact space which was ccc but did not have Property $K$. We also point out that interesting separable spaces $X(Q)$ of uncountable $\pi$-weight can be achieved in every model of set theory. In [Be88], for each regular cardinal $\kappa$ for which there exists a $\kappa$-chain of clopen sets in $\beta\omega \setminus \omega$, we produced a separable space $X(Q)$ of $\pi$-weight $\kappa$ that did not continuously map onto $I^{\omega_1}$. So Problem 2 in Shapirovskii [Sh93] has a negative answer. Referring to the last comment in this paper, it seems that the "last word" in a large part of the theory of compact spaces has not yet been spoken.

3. The Hausdorff gap space

Our example will use a $(\kappa, \kappa)$ Hausdorff gap where $\omega_1 < \kappa = cf(\kappa) \leq c$. Let $(A_\alpha, B_\alpha)_{\alpha < \kappa}$ be such that

Q1: $A_0 = \emptyset = B_0$ and $A_\alpha \cup B_\alpha \subseteq \omega$
Q2: $\alpha < \beta \Rightarrow (A_\alpha \prec * A_\beta$ and $B_\alpha \prec * B_\beta$) (strict almost inclusion)
Q3: $A_\alpha \cap B_\alpha = \emptyset$
Q4: $\not\exists A \subseteq \omega$ such that $\forall \alpha < \kappa (A_\alpha \prec \emptyset A$ and $B_\alpha \prec * \omega \setminus A)$.

Put $Q = \{ p \in P : \exists \alpha < \kappa$ with $dom(p) = * A_\alpha \cup B_\alpha$ and $p^{-1}(1) = * A_\alpha \}$ and let $X = X(Q)$. For each $q \in Q$ define $\delta(q) = \alpha < \kappa$ with $dom(q) = * A_\alpha \cup B_\alpha$ and extend $\delta$ so that $\delta : X \rightarrow \kappa$ by $\delta(I) = sup\{ \delta(q) : q \in I \}$. This definition of $\delta$ is well-defined because if $I \in X$, then by Q4, $\exists \alpha < \kappa$ such that either $A_\alpha \not\subset * \lambda(I)^{-1}(1)$ or $B_\alpha \not\subset * \lambda(I)^{-1}(0)$, hence if $\delta(q) > \alpha$, then $q /\in I$.

**Lemma 3.1.** $X$ can be partitioned into $\omega$ many closed $G_\delta$ subspaces each of which is homeomorphic to an ordinal space $[0, \alpha]$, where $|\alpha| < \kappa$. Thus, $X$ is $G_\delta$-scattered (i.e., scattered in the $G_\delta$ topology) and so cannot map continuously onto $I^{\omega_1}$.

**Proof:** Q1–Q4 allows us to easily identify, for each $f \in 2^\omega$, the closed $G_\delta$ subspace $\lambda^{-1}(f)$. If $\delta(M_f)$ is an isolated ordinal or if $\delta(M_f)$ is a limit ordinal which is not attained (i.e., $\delta(M_f) \not\in M_f$), then $\lambda^{-1}(f) \approx$ the ordinal space $[0, \delta(M_f)]$. If $\delta(M_f)$ is a limit ordinal which is attained, then $\lambda^{-1}(f) \approx [0, \delta(M_f) + 1]$. Thus, $X$ is partitioned into $\omega$ many closed $G_\delta$ ordinal subspaces and so every non-empty subspace of $X$ contains a relative $G_\delta$-point. By a result of Shapirovskii [Sh80], $X$ cannot map continuously onto $I^{\omega_1}$. \hfill $\Box$

We now partition $Q$ into horizontal sections. For each $\alpha < \kappa$ put $Q^\alpha = \{ q \in Q : \delta(q) = \alpha \}$ and put $Q^\alpha = \{ q^+ : q \in Q^\alpha \}$.

**Lemma 3.2.** For each $\alpha < \kappa$, $Q^\alpha$ is a $\pi$-base for $\{ q^+ \cap B^- : \delta(q) \leq \alpha \}$, i.e., for every $q$ and finite $B$ with $q^+ \cap B^- \not\subseteq \emptyset$ and $\delta(q) \leq \alpha$ there exists $r \in Q^\alpha$ with $r^+ \subseteq q^+ \cap B^-$.  

**Proof:** Assume $q^+ \cap B^- \not\subseteq \emptyset$ and let $\delta(q) \leq \alpha$. Choose $I \in q^+ \cap B^-$ and put $f = \lambda(I)$. Put $A = \{ p \in B : p \not\in f \}$ and $C = \{ p \in B : p \preceq f \}$. For each $p \in A$ choose $n_p \in dom(p)$ with $p(n_p) \not\in f(n_p)$. Put $R = dom(q) \cup \{ n_p : p \in A \}$
and put \( r = f \upharpoonright R \). Since \( B \cap I = \emptyset \), for each \( p \in C \) and for each finite \( H \subset \omega \) we have that \( \text{dom}(p) \not\subset R \cup H \). This implies that we can choose, for each \( p \in C \), an \( m_p \in \text{dom}(p) \) such that distinct \( p \)'s yield distinct \( m_p \)'s. Let \( s \) have domain \( \{ m_p : p \in C \} \) and satisfy that for each \( p \in C \), \( s(m_p) \neq p(m_p) \). Then, \( \delta(r \cup s) = \delta(r) = \delta(q) \leq \alpha \) and \( \emptyset \neq (r \cup s)^+ \subset q^+ \cap B^- \).

**Lemma 3.3.** \( \text{X} \) is ccc.

**Proof:** Assume not and choose an uncountable (meaning of cardinality \( \omega_1 \) throughout this proof) \( R \subset Q \) such that \( r \neq s \) in \( R \Rightarrow r \perp s \). Since \( \delta : Q \to \kappa \) is \( \leq \omega \)-to-1, choose an uncountable \( R' \subset R \) such that \( \delta \upharpoonright R' \) is 1–1. Since there exist only countably many finite collections of finite subsets of \( \omega \), choose an uncountable \( R'' \subset R' \) and finite \( F, A, G, B, H, C \subset \omega \) such that \( p \in R'' \) and \( \delta(p) = \alpha \Rightarrow \text{dom}(p) = (A_\alpha \setminus F) \cup (B_\alpha \setminus G) \cup H \) and \( p^{-1}(1) = (A_\alpha \setminus A) \cup B \cup C \) where \( F \) and \( A \) are disjoint subsets of \( A_\alpha \), \( G \) and \( B \) are disjoint subsets of \( B_\alpha \), and \( H \) and \( C \) are disjoint subsets of \( \omega \setminus (A_\alpha \cup B_\alpha) \). Let \( E = \{ \delta(p) : p \in R'' \} \).

Since \( R'' \) consists of pairwise incompatible elements, we see that for \( \alpha \neq \beta \) in \( E \), \( (A_\alpha \cup A_\beta) \cap (B_\alpha \cup B_\beta) \neq \emptyset \). Since \( \text{cf}(\kappa) > \omega_1 \), choose \( \gamma < \kappa \) such that \( \gamma > \text{sup}(E) \). Choose \( n < \omega \) and an uncountable \( K \subset E \) such that for each \( \alpha \in K \), \( A_\alpha \setminus n \subset A_\gamma \) and \( B_\alpha \setminus n \subset B_\gamma \). Since \( A_\gamma \cap B_\gamma = \emptyset \), for every \( \alpha \neq \beta \) in \( K \), \( (A_\alpha \cup A_\beta) \cap (B_\alpha \cup B_\beta) \cap n \neq \emptyset \). So we have a finite partition \( [K]_2^2 = \bigcup \{ \{ \alpha, \beta \} : i \in (A_\alpha \cup A_\beta) \cap (B_\alpha \cup B_\beta) \} \). By Ramsey’s Theorem, get \( j < n \) and \( \alpha < \beta < \eta \) in \( K \) such that \( \{ \alpha, \beta, \eta \} \) is homogeneous for \( j \). This contradicts that \( A_i \cap B_i = \emptyset \) for \( i = \alpha, \beta, \eta \). Lemma 3.3 is proved.

**Lemma 3.4.** \( Q \) is a point-\( \prec \kappa \) collection, i.e., if \( I \in X \), then \( \{ q^+ : I \in q^+ \} \) has cardinality \( \kappa \). Consequently, \( X \) does not have Property \( K_\kappa \) (\( Q \) is a collection of \( \kappa \) many clopen sets which does not have a linked subcollection of cardinality \( \kappa \)).

**Proof:** If \( I \in q^+ \) for \( \kappa \) many \( q \)'s, then \( \lambda(I)^{-1}(1) \) would fill our Hausdorff gap \( (A_\alpha, B_\alpha)_{\alpha < \kappa} \) contradicting Q4.

The above lemma tells us that \( X \) is not separable and also that \( X \) is not the support of a measure algebra (as these all have Property \( K_\kappa \) for every regular \( \kappa \)).

**Lemma 3.5.** \( X \) has countable \( \pi \)-character.

**Proof:** Let \( I \in X \) and put \( \alpha = \delta(I) \). Lemma 3.2 implies that for every neighbourhood \( q^+ \cap B^- \) of \( I \), there exists \( r^+ \in Q^\alpha \) such that \( r^+ \subset q^+ \cap B^- \). Since \( |Q^\alpha| = \omega \), we are done.

So, we have shown

**Theorem 3.6.** If there exists a \((\kappa, \kappa)\) Hausdorff gap where \( \kappa = \text{cf}(\kappa) > \omega_1 \), then there exists a compact, ccc, non-separable space \( X \) which has countable \( \pi \)-character, \( \text{character} = \sup\{ \lambda : \lambda < \kappa \} \), and which does not continuously map onto \( I^{\omega_1} \).
Corollary. MA(\(\omega_1\)) does not imply any of the following:

(a) Every compact ccc space of countable \(\pi\)-character is separable.
(b) Every compact ccc space of tightness (or even character) \(\leq \omega_1\) is separable.
(c) Every compact ccc non-separable space continuously maps onto \(I^{\omega_1}\).

Proof: We can apply the theorem because Kunen (cf. Baumgartner [Ba84]) has proved that Martin’s Axiom is consistent with \(c = \omega_2 + \) there exists a \((\omega_2, \omega_2)\) Hausdorff gap. \(\square\)

The above (a) answers the question of Juhasz [Ju77] (this question was also repeated on page 209 in Fremlin [Fr84]). The above (b) is a different kind of example showing that the theorem of Shapirovskii [Sh72]:

MA(\(\omega_1\)) \(\Rightarrow\) Every compact ccc space of tightness \(<\ \omega_1\) is separable cannot be improved in the tightness direction. It is quite different from the first published example (Bell [Be80]); that one was covered by Cantor cubes of uncountable weight. The above (c) is of interest because of the following: Let \(A\) represent the axiom of Todorcevic “Every compact ccc non-separable space maps onto \(I^{\omega_1}\)”. One use of axiom \(A\) is that it resolves several problems in the literature. S. Todorcevic has shown that \(A \Rightarrow \text{MA}(\omega_1)\). What we have shown is that \(\text{MA}(\omega_1) \not\Rightarrow A\).

In conclusion, we mention that the question of whether every model of set theory contains an example of a compact ccc non-separable space with countable \(\pi\)-character remains open.

References

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