The $G_\delta$-topology and incompactness of $\omega^\alpha$

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Abstract. We establish a relation between covering properties (e.g. Lindelöf degree) of two standard topological spaces (Lemmas 4 and 5). Some cardinal inequalities follow as easy corollaries.

Keywords: Lindelöf degree, $G_\delta$ topology, cardinal functions

Classification: 54D20, 45B10

The present note is a contribution into the study of the Lindelöf degree in powers of topological spaces. It answers a question of W.A.R. Weiss.

In what follows, $\kappa^* \subset \beta\kappa$ is the space of all free ultrafilters over a discrete set of $\kappa$ points, $\mu \kappa \subset \beta\kappa$ is the space of all uniform ultrafilters over $\kappa$, $\omega^\alpha$ denotes the $\alpha$-th power of the discrete set of integers, $(\mu \kappa, G_\delta)$ is $\mu \kappa$ with the finer $G_\delta$ topology. $L(X)$ denotes the Lindelöf degree of $X$, and $e(X) := \sup\{A \subset X : A$ is closed and discrete\} — its extent.

Kenneth Kunen ([1]) proves $L(\mu(2^\kappa)^+, G_\delta) \geq \kappa^+$, for the same $\kappa$’s as in our Corollary 6. Our Corollary 6 gives here

$L(\mu(2^\kappa)^+, G_\delta) \geq (2^\kappa)^+$.

J. Mycielski proved ([2]), by inductive “stepping up”, that, for $\alpha$ less than the $1^{st}$ weakly inaccessible cardinal,

$e(\omega^\alpha) = \alpha$.

Our Corollary 9 is a weaker statement for a larger class of cardinals. This Corollary was obtained first by Loś [3] in 1959 using group-theoretic methods. See also Juhász [4].

Stevo Todorcević ([5]) proves, assuming the combinatorial statement $\square_\kappa$,

$L(\omega^\kappa) = \kappa$. 
1. If $A = \{ A_n : n < \omega \}$ is a countable disjoint partition of the cardinal $\kappa$, then

$$\mu_\kappa = \left( \bigcup_{n<\omega} S^A_n \right) \cup \left( \tilde{S}^A \right),$$

where

$S^A_n = \{ u \in \mu_\kappa : A_n \in u \}$ and

$\tilde{S}^A = \{ u \in \mu_\kappa : \{ \bigcup_{n \geq i} A_n : i < \omega \} \subset u \}.$

Note that $\tilde{S}^A$ is a $G_\delta$ set in $\mu_\kappa$.

2. We say that a cover of $\kappa^*$ or of $\mu_\kappa$ is a proper $G_\delta$-cover if every set in it is of the form $\tilde{S}^A$ for some countable partition $A$ of $\kappa$.

3. Lemma. If $\kappa$ is a regular cardinal and $\mu_\kappa$ has a proper $G_\delta$-cover of size $\alpha$, then $\omega^\alpha$ has a subset of size $\kappa$ without a CAP (complete accumulation point).

**Proof:** Suppose $\mu_\kappa = \bigcup \{ \tilde{S}^A_\gamma : \gamma < \alpha \}$ for some collection $C = \{ A^\gamma : \gamma < \alpha \}$ of countable partitions $A^\gamma = \{ A^\gamma_n : n < \omega \}$ of $\kappa$. For every point $p \in \kappa$ define its history in $C$ $\bar{p} : \alpha \rightarrow \omega$ by setting $\bar{p}(\gamma) := n$ such that $p \in A^\gamma_n$. Let $P = \{ \bar{p} : p < \kappa \} \subset \omega^\alpha$.

**Claim 1.** $|P| = \kappa$, moreover, for every $p \in \kappa$, $K_p := \{ q \in \kappa : \bar{q} = \bar{p} \}$ has size $|K_p| < \kappa$. Indeed, if not, then no $v \ni K_p$ is covered:

$$\forall \gamma < \alpha \quad v \notin \tilde{S}^A_\gamma,$$

because

$$v \in S^A_\bar{p}(\gamma).$$

And $|P| = \kappa$ follows from the regularity of $\kappa$. \hfill \Box

**Claim 2.** $P$ has no CAP in $\omega^\alpha$. If not, let $\varphi \in \omega^\alpha$ be a CAP of $P$. Then for every finite $F \subset \alpha$

$$|\{ p < \kappa : \bar{p} \upharpoonright F = \varphi \upharpoonright F \}| = \kappa,$$

by Claim 1.

Therefore, the family $\mathcal{F} := \{ A^\gamma_{\bar{p}(\gamma)} : \gamma < \alpha \}$ has the uniform finite intersection property (i.e. $\forall \mathcal{F}_0 \in [\mathcal{F}]^{<\aleph_0} \cap \mathcal{F}_0 = \kappa$). [By $\bar{p} \upharpoonright F = \varphi \upharpoonright F \leftrightarrow p \in \bigcap_{\gamma \in F} A^\gamma_{\bar{p}(\gamma)}].$

Pick a $u \in \mu_\kappa$ extending $\mathcal{F}$.

Then $u \notin \bigcup_{\gamma < \alpha} \tilde{S}^A_\gamma = \mu_\kappa$. Contradiction. Hence $P \subset \omega^\alpha$ has no CAP in $\omega^\alpha$, so it is as required. \hfill \Box
4. Lemma. If $\kappa^*$ has a proper $G_\delta$-cover of size $\alpha$, then $\omega^\alpha$ has a closed discrete subset of size $\kappa$.

Proof: Here we, similarly, study the family $P$ of the histories of points $p \in \kappa$ in the family of partitions of $\kappa$ defining our proper $G_\delta$-cover, in this case of $\kappa^*$. Only finitely many points $p \in \kappa$ may have the same history, so $|P| = \kappa$, and, arguing as in Claim 2 of the previous lemma, $P \subset \omega^\alpha$ has no limit points in $\omega^\alpha$ whatsoever. □

5. Theorem. If $\kappa$ is a regular not Ulam measurable cardinal, then

$$L(\omega^{L(\mu\kappa,G_\delta)}) \geq \kappa.$$ 

6. Corollary. $L(\mu\kappa,G_\delta) \geq \kappa$, for the same $\kappa$'s.

Proof of Theorem 5 and Corollary 6: Since every ultrafilter over $\kappa$ is countably incomplete, there is a proper $G_\delta$ cover of $\mu\kappa$, and so $L(\mu\kappa,G_\delta) = \alpha \Rightarrow$ there is a proper $G_\delta$-cover of size $\alpha \Rightarrow$ (By Lemma 3) $\omega^\alpha$ has a subset of size $\kappa$ without a CAP

(a) $L(\omega^\alpha) \geq \kappa$, and
(b) $\alpha \geq \kappa$ (because $\alpha \geq L(\omega^\alpha))$. □

7. Corollary. $L(\omega^{2\kappa}) \geq \kappa$, for the same $\kappa$'s as in Theorem 5.

Proof: $(\mu\kappa,G_\delta)$ has a base of size $(2^\kappa)^\omega = 2^\kappa$. Hence $L(\mu\kappa,G_\delta) \leq 2^\kappa$ and so $L(\omega^{2\kappa}) \geq L(\omega^{L(\mu\kappa,G_\delta)}) \geq \kappa$. □

8. Theorem. If $\kappa$ is not Ulam measurable, then

$$L(\kappa^*,G_\delta) \geq L(\omega^{L(\mu\kappa,G_\delta)}) \geq e(\omega^{L(\kappa^*,G_\delta)}) \geq \kappa.$$ 

Proof: Immediate from Lemma 4. □

9. Corollary. If $\kappa < 1^{st}$ measurable cardinal, then

$$e(\omega^{2\kappa}) \geq \kappa,$$ 

i.e. $\omega^{2\kappa}$ has a closed discrete subspace of size $\kappa$.

Proof: Same as of Corollary 7. □

10. Corollary. Let $\lambda$ be a strong limit cardinal $\leq$ the $1^{st}$ measurable cardinal. Then the set $\{e(\omega^\alpha) : \alpha < \lambda\}$ is cofinal in $\lambda$. Hence, if $cf(\lambda) > \omega$, the set $\{\alpha < \lambda : e(\omega^\alpha) = \alpha\}$ is closed and unbounded in $\lambda$.

Remark. Murray Bell observed that the converses of Lemmas 3 and 4 are also true.

Acknowledgement. The author is very grateful to Professor William Weiss for many useful discussions.
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(Received July 19, 1995)