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On the Existence of k-SOLSSOMs

Sobre la Existencia de k-SOLSSOMs

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Abstract

In [2] Finizio reports some new results about k-SOLSSOMs. One of these results states that there exist $2-$SOLSSOM(n), for all $n > 4308$ and if $n > 135$ is an odd integer, a $2-$SOLSSOM(n) exists. In addition, when $n \equiv 0$ $(Mod\ 8)$, and $n \notin \{24, 40, 48\}$, a $2-$SOLSSOM(n) exists. These results were proved by Lee in [4].

In this paper we prove that if $p \geq 5$ is the least prime factor of n, then a $\frac{p-3}{2}-$SOLSSOM(n) exists. In particular, if $n \in \{49, 77, 91, 119, 133\}$, a $2-$SOLSSOM(n) exists, thus extending Lee's results.
Key words and phrases: Latin square, orthogonal latin square, self-orthogonal latin square, SOLSSOM.

Resumen

En [2] Finizio reporta algunos nuevos resultados acerca de k-SOLSSOMs. Uno de estos resultados afirma que para todo $n > 4308$ existe un $2-$SOLSSOM(n), y si $n > 135$ es un entero impar, existe un $2-$SOLSSOM(n). Adicionalmente, cuando $n \equiv 0$ $(Mod\ 8)$ y $n \notin \{24, 40, 48\}$, existe un $2-$SOLSSOM(n). Estos resultados fueron probados por Lee en [4].

En este artículo probamos que si $p \geq 5$ es el menor factor primo de n, entonces existe un $\frac{p-3}{2}-$SOLSSOM(n) . En particular, si $n \in \{49, 77, 91, 119, 133\}$, existe un $2-$SOLSSOM(n), extendiendo así los resultados de Lee.
Palabras y frases clave: Cuadrado latino, cuadrado latino ortogonal, cuadrado latino auto-ortogonal, SOLSSOM.

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1 Introduction

In this section we describe some definitions. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $n \times n$ matrices, the *join* (A, B) of A and B is the $n \times n$ matrix whose (i, j)-th entry is the pair (a_{ij}, b_{ij}). The latin squares A, B of order n are *orthogonal* if all the entries in the join of A and B are distinct. Latin squares $A_1, \ldots, A_r$ are *mutually orthogonal* if they are orthogonal in pairs. The abbreviation MOLS will be used for mutually orthogonal latin squares.

A *self-orthogonal* latin square (SOLS) is a latin square that is orthogonal to its transpose. Finally, a k-SOLSSOM(n) is a set $\{S_1, S_2, \ldots, S_k\}$ of self-orthogonal latin squares, together with a symmetric latin square M, for which $\{S_i, S_i^T \mid 1 \le i \le k\} \cup \{M\}$ is a set of $2k + 1$ MOLS(n).

The objective of this paper is to prove:

Theorem 1. *If $p \le 5$ is the least prime factor of n, then a $\frac{p-3}{2}$-SOLSSOM(n) exists. In particular, if $n \in \{49, 77, 91, 119, 133\}$, a 2−SOLSSOM(n) exists.*

In this paper, as usual, Z_n denotes the cyclic group of order n. Our notations are standard and taken mainly from [1] and [2].

2 Proof of the Theorem

For any odd prime power q there exist $\frac{q-3}{2}$-SOLSSOM(q) and, for $n \ge 1$, there is a $(2^{n-1} - 1)$-SOLSSOM(2^n) (see [2, 41.21]). In [4], Lee shows that there exist 2-SOLSSOM(n) for all $n > 4308$ and, if $n > 135$ is an odd integer, a 2-SOLSSOM(n) exists. In addition, when $n \equiv 0\ (Mod\ 8)$, and $n \notin \{24, 40, 48\}$, a 2-SOLSSOM(n) exists.

In this section we use a group theoretical approach to the problem of existence of k-SOLSSOMs. We first assume that $A = [a_{ij}]$ is an arbitrary latin square of order n. We define $R_i = [a_{i1} \ldots a_{in}]$ and $C_i = [a_{1i} \ldots a_{ni}]^T$, for all $1 \le i \le n$, then it is easy to see that R_i is the *i-th* row and C_i is the *i-th* column of the latin square A and we can write

$$A = [C_1 C_2 \ldots C_n] = [R_1 R_2 \ldots R_n]^T$$

Next we assume that $\sigma \in S_n$, the symmetric group on n letters, then σ induces a permutation on rows, columns and elements of the latin square A which we denote by the same symbol. Set $A_r(\sigma) = [R_{\sigma(1)} R_{\sigma(2)} \ldots R_{\sigma(n)}]^T$ and $A_c(\sigma) = [C_{\sigma(1)} C_{\sigma(2)} \ldots C_{\sigma(n)}]^T$, then it is obvious that $A_r(\sigma)$ and $A_c(\sigma)$ are latin squares and so for all $\sigma, \tau \in S_n$, $A(\sigma, \tau) = (A_r(\sigma))_c(\tau)$ is a latin square.

Now we assume that G is a group of order n and σ and τ are permutations of G which can be identified with the permutations of S_n. It is obvious that the multiplication Cayley table A of the group G is a latin square and we can use the above argument to obtain the latin square $A(\sigma, \tau)$.

We begin with an elementary lemma which will be of use later.

Lemma 1. *Let $G = \{x_1, \dots, x_n\}$ be a group of order n, A be the Cayley table of G and $\alpha, \beta, \tau, \sigma \in S_n$. The latin squares $A(\alpha, \beta)$ and $A(\tau, \sigma)$ are orthogonal if and only if, for all $1 \leq i, j, r, s \leq n$, where $(i, j) \neq (r, s)$, the following condition is satisfied:*

$$\alpha(x_i)\beta(x_j) = \alpha(x_r)\beta(x_s) \Longrightarrow \tau(x_i)\sigma(x_j) \neq \tau(x_r)\sigma(x_s).$$

Proof. Let $A = [a_{ij}]$ be the Cayley table of G, $A(\alpha, \beta) = [b_{ij}]$ and $A(\tau, \sigma) = [c_{ij}]$. Then it is easy to see that $b_{ij} = a_{\alpha(i)\beta(j)} = x_{\alpha(i)}x_{\beta(j)}$ and $c_{ij} = a_{\tau(i)\sigma(j)} = x_{\tau(i)}x_{\sigma(j)}$. Suppose $A(\alpha, \beta)$ and $A(\tau, \sigma)$ are orthogonal, $(i, j) \neq (r, s)$ and $\alpha(x_i)\beta(x_j) = \alpha(x_r)\beta(x_s)$. If $\tau(x_i)\sigma(x_j) = \tau(x_r)\sigma(x_s)$ then $b_{ij} = b_{rs}$ and $c_{ij} = c_{rs}$, hence $(b_{ij}, c_{ij}) = (b_{rs}, c_{rs})$ and by orthogonality we must have $i = r$ and $j = s$, a contradiction. Therefore the above condition is satisfied. We can repeat this argument to yield the converse of the theorem. $\square$

Corollary 1. *Suppose $\alpha, \beta \in S_n$ and let A be the multiplication table of the group G. Then the latin squares $A(i, \alpha)$ and $A(i, \beta)$, where i is the identity element of S_n, are orthogonal if and only if the map $\alpha^{-1}\beta$, where $\alpha^{-1}\beta(x) = \alpha(x)^{-1}\beta(x)$, is bijective.*

Proof. Suppose that $\alpha^{-1}\beta(x_j) = \alpha^{-1}\beta(x_s)$, so $\alpha(x_s)\alpha(x_j^{-1}) = \beta(x_s)\beta(x_j^{-1})$. If $\alpha(x_s)\alpha(x_j^{-1}) = x_r^{-1}x_i$, then $x_r\alpha(x_s) = x_i\alpha(x_j)$ and $x_r\beta(x_s) = x_i\beta(x_j)$ and by orthogonality $(i, j) = (r, s)$, i.e. $x_j = x_s$. Conversely, assume that $\alpha^{-1}\beta$ is bijective, $(i, j) \neq (r, s)$, $x_i\alpha(x_j) = x_r\alpha(x_s)$ and $x_i\beta(x_j) = x_r\beta(x_s)$. Then $\alpha^{-1}\beta(x_j) = \alpha^{-1}\beta(x_s)$ and, since $\alpha^{-1}\beta$ is bijective, $x_j = x_s$. This implies that $x_i = x_r$, which is a contradiction. $\square$

Corollary 2. *Let G be a group and $f_i \in Aut(G), 1 \leq i \leq 4$. The latin squares $A(f_1, f_2)$ and $A(f_3, f_4)$ are orthogonal if and only if for all $x, y \in G$ with $x \neq e, y \neq e$ the following condition holds:*

$$f_1(x) = f_2(y) \Longrightarrow f_3(x) \neq f_4(y)$$

Proof. We assume that $f_1(x) = f_2(y)$, so $f_1(x)f_2(e) = f_1(e)f_2(y)$, and by Lemma 1, $f_3(x)f_4(e) \neq f_3(e)f_4(y)$, i.e. $f_3(x) \neq f_4(y)$. Conversely, suppose $f_1(x_i)f_2(x_j) = f_1(x_r)f_2(x_s)$, so $f_1(x_r^{-1}x_i) = f_2(x_sx_j^{-1})$, which yields the required result. $\square$

Corollary 3. *Let G be the cyclic group of order n, p the least prime factor of n and A the Cayley table of G. Then for all $1 \leq r < p$ the latin squares $A(i, f_r)$, where $f_r(x) = rx$, are orthogonal.*

Proof. It is well known that $f_r \in Aut(Z_n)$ if and only if $(r, n) = 1$, hence for all $1 \leq i \leq p-1$, $f_i \in Aut(Z_n)$. We show now that if $1 < i, j \leq p-1$, then $f_i - f_j$ is bijective. To do this, suppose $(f_i - f_j)(x) = (f_i - f_j)(y)$. Then $(i-j)(x-y) = 0$ and since $(n, i-j) = 1$, $x = y$. Now by Corollary 1, the proof is complete. □

Remark 1. Let G be a group of order n and let A be the Cayley table of G. It is easy to see that the latin square $A(\alpha, \beta)$ is self-orthogonal if and only if $A(\alpha, \beta)$ and $A(\beta, \alpha)$ are orthogonal latin squares. Therefore by Lemma 1, the latin square $A(\alpha, \beta)$ is self-orthogonal if and only if for any elements x, y, z, t of G, the following condition holds,

$$\alpha(x)\beta(y) = \alpha(z)\beta(t) \Longrightarrow \beta(x)\alpha(y) \neq \beta(t)\alpha(z).$$

Remark 2. Assume that $f, g \in Aut(G)$. By Corollary 2, the latin square $A(f, g)$ is self-orthogonal if and only if for any non-identity elements $x, y \in G$, $f(x) = g(y)$ implies that $f(y) \neq g(x)$.

Remark 3. Let G be the cyclic group of order n and A be the Cayley table of G. If p is the least prime factor of n then all of the latin squares $A(i, f_r), 1 < r < p-1$, are mutually and self-orthogonal.

Proof of Theorem 1. Assume that there exists a $2 \times k$ matrix $M = (a_{ij})$ with the following conditions :

(a) For all $i = 1, 2$ and $1 \leq j \leq n$, $a_{ij} < n$, $(a_{ij}, n) = 1$ and $a_{1j} \neq a_{2j}$,
(b) For all $1 \leq j' \leq j \leq k$, $((a_{1j}a_{1j'} - a_{2j}a_{2j'}), n) = 1$,
(c) For all $1 \leq j' \leq j \leq k$, $((a_{1j}a_{2j'} - a_{2j}a_{1j'}), n) = 1$.

We now show that with these conditions, a k$-$SOLSSOM(n) exists. To do this, let $i = 1, 2, 1 \leq j \leq k$ and g_{ij} be the map $x \mapsto a_{ij}x$. Then it is easy to see that $g_{ij} \in Aut(Z_n)$. Set $A_j = A(g_{1j}, g_{2j})$, in which A is the latin square obtained from the Cayley table for Z_n. Then we can see that the set $\{A_j, A_j^t \mid 1 \leq j \leq k\} \cup \{A\}$ form a k$-$SOLSSOM(n). Suppose that $p \geq 5$ is the least prime factor of n. We now define a $2 \times \frac{p-3}{2}$ matrix $M = (a_{ij})$ as follows:

$$a_{1j} = j \qquad \text{and} \qquad a_{2j} = j + 1.$$

Now we can see that the matrix M satisfies the conditions (a), (b) and (c). Therefore a $\frac{p-3}{2}$$-$SOLSSOM($n$) exists. □

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References

[1] I. Anderson, *Combinatorial Designs: Construction Methods*, Ellis Horwood Series in Mathematics and its Applications, John Wiley & Sons, 1990.

[2] N. J. Finizio, *SOLS with a symmetric orthogonal mate (SOLSSOM)*, in: C. J. Colbourn, J. H. Dinitz (eds.) *The CRC Handbook of Combinatorial Designs*, CRC Press, 1995, 447–452.

[3] N. J. Finizio, *Some quick and easy SOLSSOMs*, Congr. Numer. **99** (1994), 307–313.

[4] T. C. Y. Lee, *Tools for constructing RBIBDs, Frames, NRBs and SOLSSOMs*, Ph.D. thesis, University of Waterloo, 1995.