# Moore graphs and beyond: A survey of the degree/diameter problem

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**Abstract**

The degree/diameter problem is to determine the largest graphs or digraphs of given maximum degree and given diameter. General upper bounds – called Moore bounds – for the order of such graphs and digraphs are attainable only for certain special graphs and digraphs. Finding better (tighter) upper bounds for the maximum possible number of vertices, given the other two parameters, and thus attacking the degree/diameter problem 'from above', remains a largely unexplored area. Constructions producing large graphs and digraphs of given degree and diameter represent a way of attacking the degree/diameter problem 'from below'. This survey aims to give an overview of the current state-of-the-art of the degree/diameter problem. We focus mainly on the above two streams of research. However, we could not resist mentioning also results on various related problems. These include considering Moore-like bounds for special types of graphs and digraphs, such as vertex-transitive, Cayley, planar, bipartite, and many others, on the one hand, and related properties such as connectivity, regularity, and surface embeddability, on the other hand.

# Contents

# 1 Introduction

The topology of a network (such as a telecommunications, multiprocessor, or local area network, to name just a few) is usually modelled by a graph in which vertices represent 'nodes' (stations or processors) while undirected or directed edges stand for 'links' or other types of connections.

In the design of such networks, there are a number of features that must be taken into account. The most common ones, however, seem to be limitations on the vertex degrees and on the diameter. The network interpretation of the two parameters is obvious: The degree of a vertex is the number of connections attached to a node, while the diameter indicates the largest number of links that must be traversed in order to transmit a message between any two nodes.

What is then the largest number of nodes in a network with a limited degree and diameter? If links are modelled by undirected edges, this leads to the

- *Degree/Diameter Problem*: Given natural numbers $\Delta$ and $D$, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree $\Delta$ and diameter $\leq D$.

The statement of the directed version of the problem differs only in that 'degree' is replaced by 'out-degree'. We recall that the out-degree of a vertex in a digraph is the number of directed edges leaving the vertex. We thus arrive at the

- *(Directed) Degree/Diameter Problem*: Given natural numbers $d$ and $k$, find the largest possible number of vertices $n_{d,k}$ in a digraph of maximum out-degree $d$ and diameter $\leq k$.

Research activities related to the degree/diameter problem fall into two main streams. On the one hand, there are proofs of non-existence of graphs or digraphs of order close to the general upper bounds, known as the Moore bounds. On the other hand, there is a great deal of activity in the constructions of large graphs or digraphs, furnishing better lower bounds on $n_{\Delta,D}$ (resp., $n_{d,k}$).Since the treatments of the undirected and directed cases have been quite different, we divide further exposition into two parts. Part 1 deals with the undirected case and Part 2 with the directed one.

We first discuss the existence of Moore graphs (Section 2.1) and Moore digraphs (Section 3.1). These are graphs and digraphs which attain the so called Moore bound, giving the theoretical maximum for the order of a graphs (resp., digraph) given diameter and maximum degree (resp., out-degree).

Then we present known results on the existence of graphs (Section 2.2) and digraphs (Section 3.2) whose order is 'close' to the Moore bound, whenever the Moore bound cannot be attained.

The question of regularity of graphs close to the Moore bound is much more interesting for directed graphs than for undirected ones, and so we include a section (3.3) on this topic only in Part 2.

The next two sections (2.3 and 3.4) are then devoted to the constructions of large graphs and digraphs. In Sections 2.4 and 3.5 we introduce and discuss several restricted versions of the degree/diameter problem for graphs and digraphs.

Various related topics are listed in Sections 2.5 and 3.6. Finally, in the Conclusion, we present a short list of some of the interesting open problems in the area.

# 2 Part 1: Undirected graphs

## 2.1 Moore graphs

There is a straightforward upper bound on the largest possible order (i.e., the number of vertices) $n_{\Delta,D}$ of a graph $G$ of maximum degree $\Delta$ and diameter $D$. Trivially, if $\Delta = 1$ then $D = 1$ and $n_{1,1} = 2$; in what follows we therefore assume that $\Delta \geq 2$.

Let $v$ be a vertex of the graph $G$ and let $n_i$, for $0 \leq i \leq D$, be the number of vertices at distance $i$ from $v$. Since a vertex at distance $i \geq 1$ from $v$ can be adjacent to at most $\Delta - 1$ vertices at distance $i+1$ from $v$, we have $n_{i+1} \leq (\Delta - 1)n_i$, for all $i$ such that $1 \leq i \leq D-1$. With the help of $n_1 \leq \Delta$, it follows that $n_i \leq \Delta(\Delta-1)^{i-1}$, for $1 \leq i \leq D$. Therefore,

$$\begin{aligned} n_{\Delta,D} = \sum_{i=0}^{D} n_i &\leq 1 + \Delta + \Delta(\Delta-1) + \ldots + \Delta(\Delta-1)^{D-1} \\ &= 1 + \Delta(1 + (\Delta-1) + \ldots + (\Delta-1)^{D-1}) \\ &= \begin{cases} 1 + \Delta\,\frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta > 2 \\ 2D+1 & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{1}$$

The right-hand side of (1) is called the *Moore bound* and is denoted by $M_{\Delta,D}$. The bound was named after E. F. Moore who first proposed the problem, as mentioned in [160]. A graph whose order is equal to the Moore bound $M_{\Delta,D}$ is called a *Moore graph*; such a graph is necessarily regular of degree $\Delta$.

The study of Moore graphs was initiated by Hoffman and Singleton. Their pioneering paper [160] was devoted to Moore graphs of diameter 2 and 3. In the case of diameter $D = 2$, they proved that Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57 but for no other degrees, and that for the first three values of $\Delta$ the graphs are unique. For $D = 3$ they showed that the unique Moore graph is the heptagon (for $\Delta = 2$). The proofs exploit eigenvalues and eigenvectors of the adjacency matrix (and its principal submatrices) of graphs.

It turns out that no Moore graphs exist for the parameters $\Delta \geq 3$ and $D \geq 3$. This was shown by Damerell [95] by way of an application of his theory of distance-regular graphs to the classification of Moore graphs. An independent proof of this result was also given by Bannai and Ito [19].

Main results concerning Moore graphs can therefore be summed up as follows. Moore graphs for diameter $D = 1$ and degree $\Delta \geq 1$ are the complete graphs $K_{\Delta+1}$. For diameter $D = 2$, Moore graphs are the cycle $C_5$ for degree $\Delta = 2$, the Petersen graph (see Fig. 1) for degree $\Delta = 3$, and the Hoffman-Singleton graph (see Fig. 2, drawn by Slamin [220]) for degree $\Delta = 7$. Finally, for diameter $D \geq 3$ and degree $\Delta = 2$, Moore graphs are the cycles on $2D+1$ vertices $C_{2D+1}$.

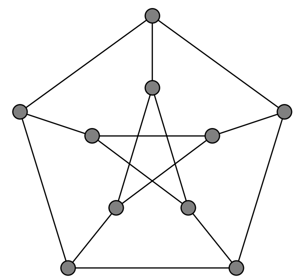

Figure 1: Petersen graph.

Between the time of the publication of the Hoffman-Singleton paper (1960) and the time of the publication of the results by Bannai-Ito and Damerell (both in 1973), there were several related partial results published. For example, Friedman [134] showed that there are no Moore graphs for parameters $(\Delta, D)$, where $\Delta = 3, 4, 5, 6, 8$ and $3 < D \leq 300$, except, possibly, for the pair $(5, 7)$. He also showed that there are no Moore graphs with parameters $(3, D)$, when $D \geq 3$ and $2D + 1$ is prime. Bosák [60] proved the nonexistence of Moore graphs of degree 3 and diameter $D$, $3 \leq D \leq 8$. For another contribution to nonexistence proofs, see also Plesník [202]. A combinatorial proof that the Moore graph with $\Delta = 7$ and $D = 2$ is unique was given by James [167]. As an aside, a connection of Moore graphs with design theory was found by Benson and Losey [38], by embedding the Hoffman-Singleton graph in the projective plane $PG(2, 5^2)$.

Several other areas of research in graph theory turn out to be related or inspired by the theory of Moore graphs; examples include cages, antipodal graphs, Moore geometries, and Moore groups. Recall that a $(k, g)$-*cage* is a graph of degree $k$ and girth $g$, with the minimum possible number of vertices. Connections between cages and Moore graphs are explained in a survey paper on cages by Wong [236].

A graph is *antipodal* if for each vertex $x$ there exists a vertex $z$ such that $d(x, y) + d(y, z) = d(x, z)$, for all vertices $y$ of the graph. Sabidussi [208] showed that Moore graphs of diameter 2 and degree 3, 7, or possibly 57 are 'antipodal quotients' of certain extremal antipodal graphs of odd diameter.

Fuglister [135, 136] and Bose and Dowling [64] investigated finite Moore geometries which are a generalisation of Moore graphs; other contribution to this area include Damerell and Georgiacodis [97], Damerell [96], and Roos and van Zanten [206, 207]. Finally, Fried and Smith [132] defined a Moore group and proved results that limit the possible degrees that Moore groups of fixed rank can have, by reducing the problem to the study of Moore graphs.

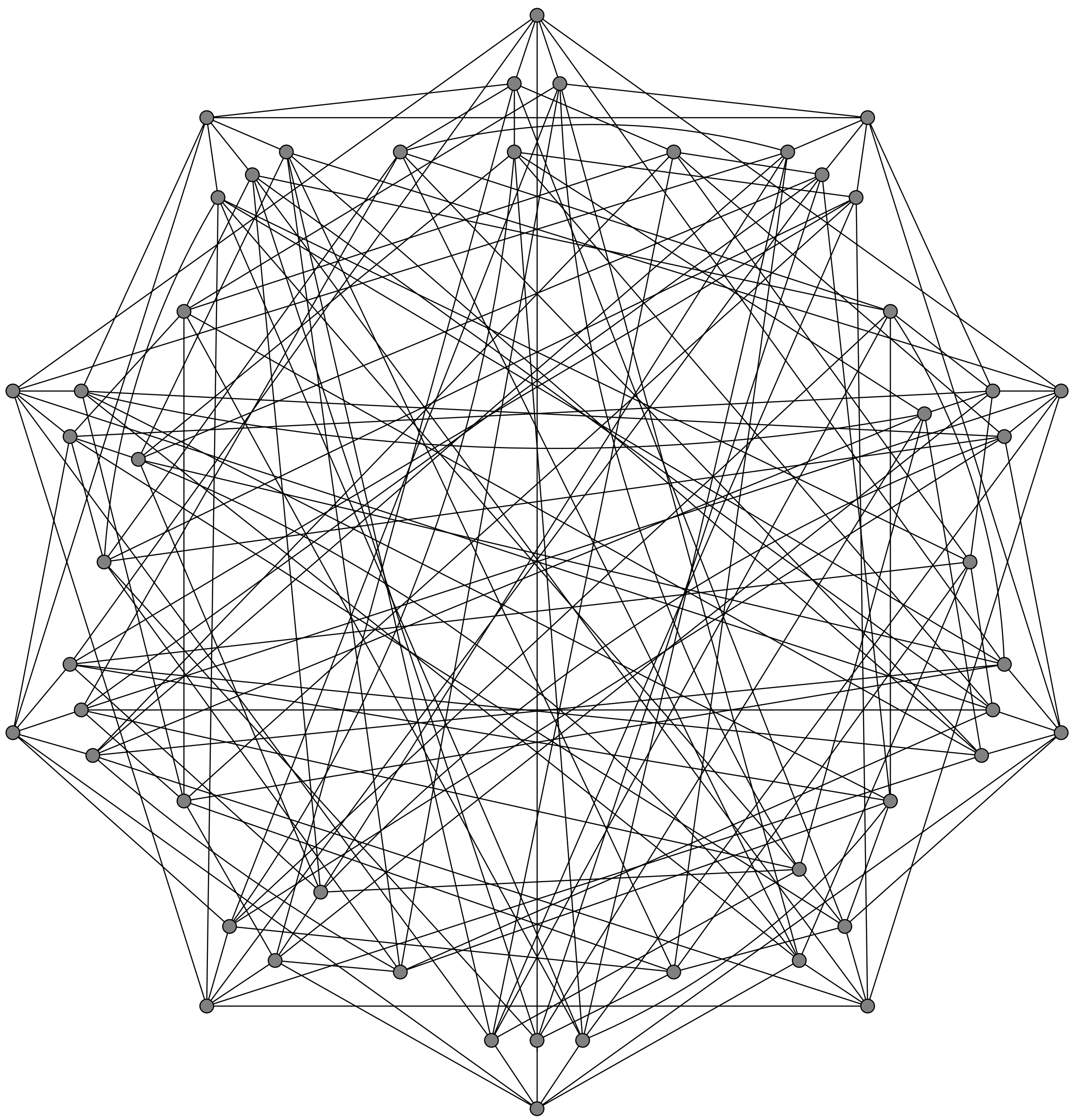

Figure 2: Hoffman-Singleton graph.

## 2.2 Graphs of order close to Moore bound

Since Moore graphs exist only in a small number of cases, the study of the existence of large graphs of given diameter and maximum degree focuses on graphs whose order is 'close' to the Moore bound, that is, graphs of order $M_{\Delta,D}-\delta$, for $\delta$ small. The parameter $\delta$ is called the *defect*, and the most usual understanding of 'small defect' is that $\delta \leq \Delta$.

For convenience, by a $(\Delta, D)$-graph we will understand any graph of maximum degree $\Delta$ and of diameter at most $D$; if such a graph has order $M_{\Delta,D}-\delta$ then it will be referred to as a $(\Delta, D)$-graph of defect $\delta$.

Erdős, Fajtlowitcz and Hoffman [111] proved that, apart from the cycle $C_4$, there are no graphs of degree $\Delta$, diameter 2 and defect 1, that is, of order one less than the Moore bound; for a related result, see Fajtlowicz [117].

This was subsequently generalized by Bannai and Ito [20], and also by Kurosawa and Tsujii [176], to all diameters. Hence, for all $\Delta \geq 3$, there are no $(\Delta, D)$-graphs of defect 1, and for $\Delta = 2$ the only such graphs are the cycles $C_{2D}$. It follows that, for $\Delta \geq 3$, we have $n_{\Delta,D} \leq M_{\Delta,D} - 2$.

Let us now discuss the case of defect $\delta = 2$. Clearly, if $\Delta = 2$ then the $(\Delta, D)$-graphs of defect 2 are the cycles $C_{2D-1}$. For $\Delta \geq 3$, only five $(\Delta, D)$-graphs of defect 2 are known at present: Two $(3, 2)$-graphs of order 8, a $(4, 2)$-graph of order 15, a $(5, 2)$-graph of order 24, and a $(3, 3)$-graph of order 20.

The last three of these graphs, which are depicted in Fig. 3, were found by Elspas [109] and are known to be unique; in Bermond, Delorme and Farhi [42], the $(3, 3)$-graph was constructed as a certain product of a 5-cycle with the field of order four. These results (together with [20]) imply that $n_{4,2} = 15$, $n_{5,2} = 24$, and $n_{3,3} = 20$.

In [194, 195], ed and Miller proved some structural properties of graphs of diameter 2 and show that graphs of diameter 2 and defect 2 do not exist for many values of $d$.

Little is known about graphs with defects larger than two. Jorgensen [168] proved that a graph with maximum degree 3 and diameter $D \geq 4$ cannot have defect 2, which shows that $n_{3,D} \leq M_{3,D} - 3$ if $D \geq 4$; for $D$ equal to 4 this was previously proved by Stanton, Seah and Cowan [228].

Recently, Miller and Simanjuntak [189] proved that a graph with maximum degree 4 and diameter $D \geq 3$ cannot have defect 2 which shows that $n_{4,D} \leq M_{4,D} - 3$ if $D \geq 3$. Some further upper bounds on the maximum number of vertices for graphs which are not Moore were given by Smyth [224]. See also Buskens and Stanton [70], Buskens, Rogers and Stanton [71], and Cerf, Cowan, Mullin and Stanton [79, 80] for work related to (small) graphs of order close to Moore bound.

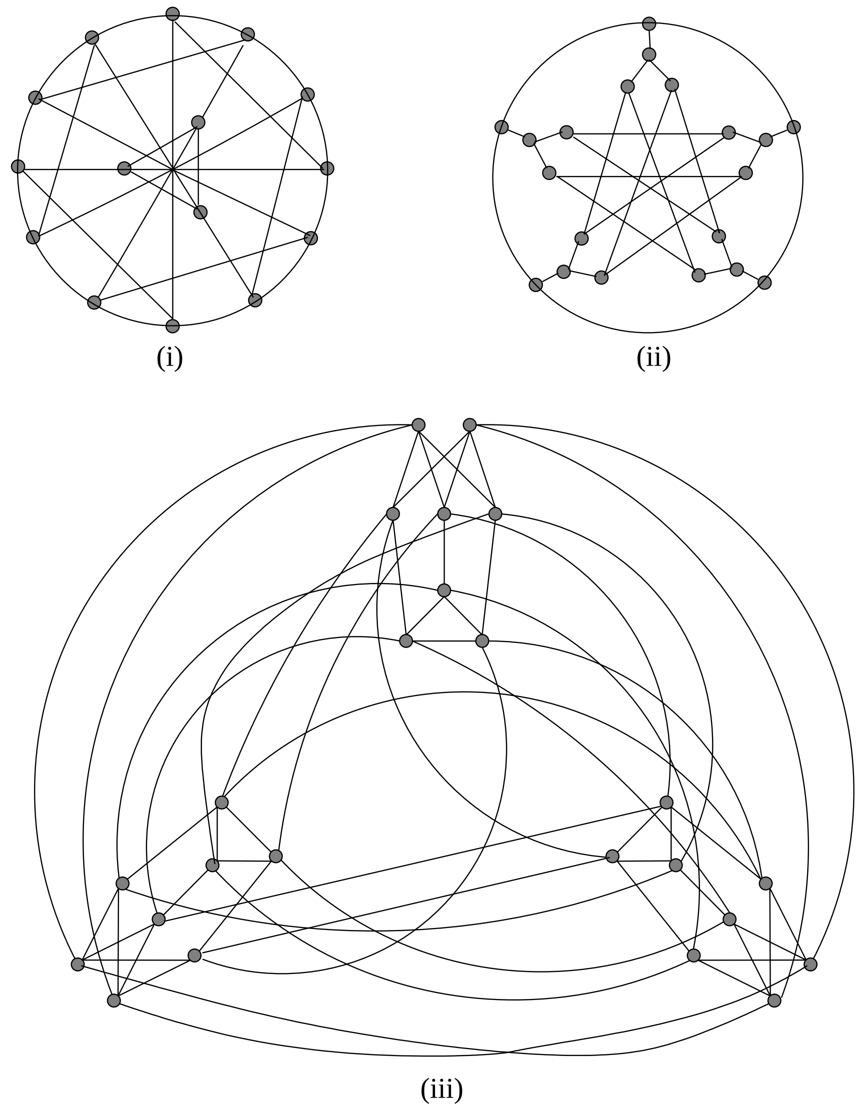


Figure 3: Examples of graphs of order $M_{\delta,D} - 2$ :
(i) $n = 15$, $\Delta = 4$, $D = 2$; (ii) $n = 20$, $\Delta = 3$, $D = 3$; (iii) $n = 24$, $\Delta = 5$, $D = 2$.

In [193], Nguyen and Miller proved some structural properties of graphs of diameter 2 and maximal repeats, that is, graphs with the property that there exists a vertex with unique paths of lengths at most the diameter to all the other vertices except one vertex to which the number of walks of length at most the diameter is equal exactly to the defect plus 1. Furthermore, in [195], they considered graphs with diameter 2 and defect 3. They proved that such graphs must contain a certain induced subgraph, which in turn leads to the proof that, for degree 6 and diameter 2, the largest order of a vertex-transitive graph is 32.

We summarise our current knowledge about the upper bound on the order of graphs of degree $\Delta$ and diameter $D$ in Table 1.

| *Diameter* $D$ | *Maximum Degree* $\Delta$ | *Upper Bound for Order* $n_{\Delta,D}$ |
|---|---|---|
| $D = 1$ | $\Delta \geq 1$ | $M_{\Delta,1}$ |
| $D = 2$ | $\Delta = 2, 3, 7,\ 57(?)$ | $M_{\Delta,2}$ |
| | other $\Delta \geq 2$ | $M_{\Delta,2} - 2$ |
| $D = 3$ | $\Delta = 2$ | $M_{2,3}$ |
| | $\Delta = 3$ | $M_{3,3} - 2$ |
| | $\Delta = 4$ | $M_{4,D} - 3$ |
| | all $\Delta \geq 5$ | $M_{\Delta,3} - 2$ |
| $D \geq 4$ | $\Delta = 2$ | $M_{2,4}$ |
| | $\Delta = 3$ | $M_{3,D} - 3$ |
| | $\Delta = 4$ | $M_{4,D} - 3$ |
| | all $\Delta \geq 5$ | $M_{\Delta,D} - 2$ |

Table 1: Current upper bounds of $n_{\Delta,D}$.

In Sections 2.1 and 2.2, we have seen that for certain pairs $(\Delta, D)$ there exist graphs of order close to (and in some cases equal to) the Moore bound. The situation for pairs $(\Delta, D)$ not discussed above is largely unknown. In this connection, Bermond and Bollobás [39] asked the following interesting question: *Is it true that for each positive integer c there exist* $\Delta$ *and* $D$ *such that the order of the largest graph of maximum degree* $\Delta$ *and diameter* $D$ *is at most* $M_{\Delta,D} - c$ *?*

## 2.3 Constructions of large graphs

Another way to study graphs close to the Moore bound is by constructing large graphs in order to find improvements in the lower bound on the maximum possible order of graphs for given $D$ and $\Delta$. This has been done in various ways, and often by considering particular classes of graphs, such as vertex-transitive and Cayley graphs (which will be discussed in more detail in forthcoming sections).

### 2.3.1 General overview

The *undirected de Bruijn graph* of type $(t, k)$ has vertex set $V$ formed by all sequences of length $k$, the entries of which are taken from a fixed alphabet consisting of $t$ distinct letters. In the graph, two vertices $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are joined by an edge if either $a_i = b_{i+1}$ for $1 \leq i \leq k-1$, or if $a_{i+1} = b_i$, for $1 \leq i \leq k-1$. Obviously, the undirected deBruijn graph of type $(t, k)$ has order $t^k$, degree $\Delta = 2t$ and diameter $D = k$. These graphs therefore give, for any $\Delta$ and $D$, the lower bound

$$n_{\Delta,D} \geq \left(\frac{\Delta}{2}\right)^D .$$

Various improvements on this bound have been obtained. For example, ignoring directions in the digraph construction of Baskoro and Miller [28] produces graphs of even maximum degree $\Delta$ and diameter at most $D$; the order of these graphs is

$$\left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}.$$

A substantial progress was recently achieved by Canale and Gómez [75] by exhibiting, for an infinite set of values of $\Delta$, families of graphs showing that

$$n_{\Delta,D} \geq \left(\frac{\Delta}{1.57}\right)^D$$

for $D$ congruent with $-1$, 0, or 1 (mod 6).

For completeness, we mention several related results. Certain extensions of deBruijn graphs were studied by Canale and Gómez [76]. An adaptation of the digraph construction of Imase and Itoh [163] also gives $(\Delta, D)$-graphs of order at least $\lceil \frac{\Delta}{2} \rceil^D$. The list of general lower bounds also includes constructions by Elspas [109], Friedman [133], Korn [174], Akers [5] and Arden and Lee [11], all giving $(\Delta, D)$-graphs of order

$$f(\Delta)(\Delta - 1)^{\lceil \frac{D}{2} \rceil} + g(\Delta),$$

where $f$ and $g$ depend on $\Delta$ but not on $D$.

Much better results have been obtained for small values of $D$. By far the best result is furnished by Brown's construction [68], with the help of finite projective geometries. Let

$q$ be a prime power and let $F$ be the Galois field of order $q$. A *projective point* is any collection of $q-1$ triples of the form $(ta, tb, tc)$, where $t \in F$ and $(a, b, c)$ is a non-zero vector in $F^3$; any non-zero triple in this set is a *representative* of the point. Let $\mathcal{P}$ be the set of all such points; it is easy to see that $|\mathcal{P}| = q^2 + q + 1$. Let $G$ be the graph with vertex set $\mathcal{P}$, where two vertices are adjacent if the corresponding projective points have orthogonal representatives. Since any two non-orthogonal representatives are orthogonal to some non-zero element of $F^3$, the graph $G$ has diameter 2. In general, $G$ need not be regular but its maximum degree is always $\Delta = q + 1$. Therefore, for each $\Delta$ such that $\Delta - 1$ is a prime power, we have [68]

$$n_{\Delta,2} \geq \Delta^2 - \Delta + 1. \tag{2}$$

As observed by Erdős, Fajtlowicz and Hoffman [111], and by Delorme [100], this bound can be improved to

$$n_{\Delta,2} \geq \Delta^2 - \Delta + 2 \tag{3}$$

if $\Delta - 1$ is a power of 2. In order to obtain a lower bound for the remaining values of $\Delta$, we may use the following fact [162] about the distribution of prime numbers: For an arbitrary $\varepsilon > 0$, there is a constant $b_\varepsilon$ such that for any natural $m$ there is a prime between $m$ and $b_\varepsilon m^{7/12+\varepsilon}$. This, in combination with vertex duplication (insertion of new vertices adjacent to all neighbours of certain old vertices) in the graphs of [68], implies that for any $\varepsilon > 0$ there is a constant $c_\varepsilon$ such that, for any $\Delta$, we have

$$n_{\Delta,2} \geq \Delta^2 - c_\varepsilon \Delta^{19/12+\varepsilon}. \tag{4}$$

For larger diameter, it seems more reasonable to focus on asymptotic behaviour of $n_{\Delta,D}$ for fixed $D$ while $\Delta \to \infty$. Delorme [99] introduced the parameter

$$\mu_D = \lim\ \inf_{\Delta\to\infty} \frac{n_{\Delta,D}}{\Delta^D}.$$

Trivially, $\mu_D \leq 1$ for all $D$, and $\mu_1 = 1$; the bound (4) shows that $\mu_2 = 1$ as well. Further results of Delorme [98] imply that $\mu_D$ is equal to 1 also for $D = 3$ and $D = 5$.

The above facts can be seen as an evidence in favour of an earlier conjecture of Bollobás [57] that, for each $\varepsilon > 0$, it should be the case that

$$n_{\Delta,D} > (1 - \varepsilon)\Delta^D$$

if $\Delta$ and $D$ are sufficiently large.

The values of $\mu_D$ for other diameters $D$ are unknown. For example, for diameter 4 we only know that $\mu_4 \geq 1/4$; see Delorme [100] for more information.

### 2.3.2 Star product and compounding

A number of sophisticated constructions arose in the quest for large graphs of given degree and diameter. We comment in some detail on two that seem to be most important: the star product of Bermond, Delorme and Farhi [41, 42] and the compounding of graphs introduced by Bermond, Delorme and Quisquater [43].

The concept of the *star product* of two graphs $H$ and $K$ was introduced by Bermond, Delorme and Farhi [41] as follows. Fix an arbitrary orientation of all edges of $H$ and let $\vec{E}$ be the corresponding set of the fixed darts of $H$. For each dart $uv \in \vec{E}$, let $\phi_{uv}$ be a bijection on the set $V(K)$. Then the vertex set of the star product $H * K$ is $V(H) \times V(K)$, and a vertex $(u, k)$ is joined in $H * K$ to a vertex $(v, l)$ if and only if either $u = v$ and $kl$ is an edge of $K$, or if $uv \in \vec{E}$ and $l = \phi_{uv}(k)$.

Loosely speaking, the star product of $H$ and $K$ can be formed by taking $|V(H)|$ copies of $K$, whereby two copies of $K$ 'represented' by vertices $u, v \in V(H)$ are interconnected by a perfect matching (that depends on the bijection $\phi_{uv}$), whenever $uv \in \vec{E}$.

With the help of the star product, Bermond, Delorme and Farhi [41, 42] described several families of large $(\Delta, D)$-graphs for various values of $\Delta$ and $D$. An inspection of their examples reveals, however, that in *all* instances they actually used a special case of the star product that we describe next.

Let $\Gamma$ be a group and let $S$ be a symmetric unit-free generating set $S$, meaning that $S^{-1} = S$ and $1_\Gamma \neq S$. The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set $\Gamma$, two vertices $a, b$ being adjacent if $a^{-1}b \in S$. In the above definition of the $*$-product $H * K$, take now $K = C(\Gamma, S)$ and $\phi_{uv}(k) = g_{uv}\psi_{uv}(k)$, where $g_{uv}$ is an arbitrary element of $\Gamma$ and $\psi_{uv}$ is an automorphism of $\Gamma$. In [41, 42], the authors used this special version of the $*$-product mainly with Cayley graphs of cyclic groups and of the additive groups of finite fields. We now briefly comment on compounding. Roughly speaking, compounding of

two graphs $G$ and $H$ is obtained by taking $|V(H)|$ copies of $G$, indexed by the vertices of $H$, and joining two copies $G_u$, $G_v$ of $G$ by a single edge (or a pair of edges) whenever $uv$ is an edge of $H$. Depending on particular positions of edges between copies of the graph $G$, one may obtain various large graphs of given degree and diameter.

This method tends to give good results, especially in *ad hoc* combinations with other methods. For instance, Fiol and Fábrega [124], and Gómez [141], considered compounding combined with graphs on alphabets, where vertices are words over a certain alphabet and adjacency is defined by various relations between words. Large graphs of diameter 6, obtained by methods in this category, were given by Gómez [142]. Other related results were produced by Fiol, Yebra and Fábrega [131], and by Gómez and Fiol [146].

Several other *ad hoc* methods have been designed in connection with searching for large

$(\Delta, D)$-graphs for relatively small values of $\Delta$ and $D$. As most of these methods are based on graphs related in one way or another to algebraic structures (mostly groups), we will discuss them in more detail in the next subsection. Here we just mention a method by Gómez, Pelayo and Balbuena [152] that produces large graphs of diameter six by replacing some vertices of a Moore bipartite graph of diameter six with graphs $K_h$ which are joined to each other and to the rest of the graph using a special graph of diameter two. The degree of the constructed graph remains the same as the degree of the original graph. In an extension to this work, Gómez and Miller [149] presented two new generalizations of two large compound graphs.

### 1.3.3 Graph lifting

Graph lifting has been well known in algebraic and topological graph theory for decades [153]. It is well suited for producing large $(\Delta, D)$-graphs since a number of other construction methods can be reduced to lifting. In order to describe the graph lifting construction, we will think of (undirected) edges as being formed by pairs of oppositely directed *darts*; if $e$ is a dart then $e^{-1}$ will denote its reverse. The set $D(G)$ of all darts of $G$ then satisfies $|D(G)| = 2|E(G)|$. For a finite group $\Gamma$, a mapping $\alpha: D(G) \to \Gamma$ will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any dart $e \in D(G)$. The pair $(G, \alpha)$ determines the *lift* $G^\alpha$ of $G$. The vertex set and the dart set of the lift are $V(G^\alpha) = V(G) \times \Gamma$ and $D(G^\alpha) = D(G) \times \Gamma$, In the lift, $(e, g)$ is a dart from the vertex $(u, g)$ to the vertex $(v, h)$ if and only if $e$ is a dart from $u$ to $v$ in the *base graph* $G$ and, at the same time, $h = g\alpha(e)$. The lift is an undirected graph because the darts $(e, g)$ and $(e^{-1}, g\alpha(e))$ are mutually reverse and form an undirected edge of $G^\alpha$.

Figure 4 shows an example of a base graph with ordinary voltages in the group $\mathcal{Z}_5 \times \mathcal{Z}_5$ which lifts to the Hoffman-Singleton graph, displayed in Fig. 2; the function $p(i)$ in Fig. 4 can be any quadratic polynomial over $\mathcal{Z}_5$ in the variable $i$, as follows from [213].

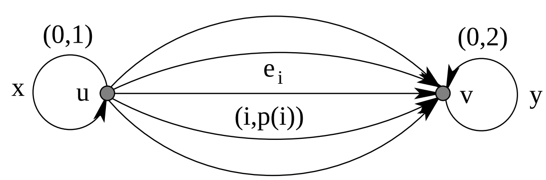


Figure 4: The base graph for the Hoffman-Singleton graph.

It is known [153] that a graph $H$ is a lift (of a smaller graph) if and only if the automorphism group of $H$ contains a non-trivial subgroup acting freely on the vertex set of $H$. This condition is in fact satisfied for most of the current largest examples of $(\Delta, D)$-graphs, and hence most of these can be described as lifts.

The latest examples of reformulating an existing construction in terms of lifts are the largest known $(\Delta, D)$-graphs for the pairs (3,7), (3,8), (4,4), (5,3), (5,5), (6,3), (6,4), (7,3), (14,3), and (16,2), initially obtained by Exoo [112] by computer search. Having mentioned computers, we note that the diameter of the lift can be conveniently expressed in terms of voltages on walks of the base graph [65]; besides its theoretical importance, this fact can be used to design efficient diameter-checking algorithms.

The cases when the base graphs are bouquets of loops (possibly with semi-edges, i.e., 'dangling' non-loop edges incident with just one vertex) are of particular importance, since their lifts are Cayley graphs. A more colloquial but equivalent definition of a Cayley graph was given in the previous subsection. While Cayley graphs are always lifts of single-vertex graphs, in many instances quite complex Cayley graphs (such as the Cayley graphs of certain semidirect products of Abelian groups considered in [105]) can actually be described as ordinary lifts of smaller Cayley graphs, with voltages in Abelian (mostly cyclic) groups; see [66].

For more general base graphs, there exist convenient sufficient conditions [66] for a lift to be a vertex transitive (or a Cayley) graph, which can be successfully used to produce large vertex transitive $(\Delta, D)$-graphs by lifts. Results for vertex-transitive and Cayley graphs will be surveyed in Subsections 2.4.1 and 2.4.2. We conclude this subsection with a remark relating lifts of graphs with the $*$-product $G * H$: If $H$ is a Cayley graph and if the group values on the edges of $G$ are taken in the Cayley group of $H$ then $G * H$ is just a lift of $G$.

### 2.3.3 Tables of large graphs

Needless to say that in many cases the largest currently known $(\Delta, D)$-graphs have been found with the assistance of computers. It is clear that computation of diameter is much easier in the case of graphs that admit a lot of symmetries; here of particular advantage are vertex-transitive graphs which we will discuss in the next subsection. At this point we note that Toueg and Steiglitz [232] present a local search algorithm for the design of small diameter networks, for both directed and undirected graphs. The resulting graphs tend to have small diameter and small average shortest distance.

Descriptions of many new constructions, often accompanied by a new corresponding table of the largest known values of $(\Delta, D)$-graphs, are published frequently. These include constructions by Alegre, Fiol and Yebra [8], Bar-Yehuda and Etzion [22], Bermond, Delorme and Farhi [41, 42], Bermond, Delorme and Quisquater [44, 45, 47, 46], Campbell *et al.* [74], Carlsson, Cruthirds, Sexton and Wright [78], Chudnovsky, Chudnovsky and Denneau [82], Chung [83], Comellas and Gómez [93], Delorme [98, 100], Delorme and Farhi [101], Dinneen and Hafner [105], Doty [106], Gómez, Fiol and Serra [147], Gómez, Fiol and Yebra [148], Hafner [155], Memmi and Raillard [180], Smyth [223], and Storwick [229].

Table 2 shows a summary of current largest known graphs for degree $\Delta \leq 16$ and diameter $D \leq 10$. These graphs provide the best current lower bounds on the order of graphs for given values of degree and diameter. This table can be found on the website

"http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.html"

which is updated regularly by Francesc Comellas. A latex file of this table can be obtained upon request from Charles Delorme at email "cd@lri.fr".

Recent updates in Table 2 are due to Exoo: entries (3,6)-(3,8), (4,4), (4,7), (5,3), (5,5), (6,3), (6,4), (7,3), (16,2); to Hafner: entries (5,9), (5,10), (6,7)-(6,10), (7,6)-(7,10), (8,5), (8,7), (8,9), (8,10), (9,7), (9,10), (10,5), (10,7)-(10,10), (11,5), (11,7), (11,8), (12,7), (13,5), (13,7), (13,8), (14,5), (14,8), (15,8); to Quisquater: entries (3,9), (3,10); to Gómez and Pelayo: entries (5,6), (6,6), (8,6), (9,6), (10,6), (12,6), (14,9); to Sampels: entries (4,8), (4,10), (5,8)-(5,10), (6,7)-(6,10), (7,6)-(7,10), (8,8)-(8,10), (9,4), (9,5), (9,8)-(9,10), (10,5), (10,7), (10,8)-(10,10); to McKay, Miller, Siráň: entries (11,2), (13,2); and to Gómez: entries (5,6), (8,6), (9,6), (10,6), (12,6), (14,6) [89].

## 2.4 Restricted versions of the degree/diameter problem

The study of large graphs of given degree and diameter has often been restricted to special classes of graphs. The most obvious candidates here are vertex-transitive and Cayley graphs, suitable because of their quick computer generation as well as from the point of view of diameter checking. Other special classes, for which the degree/diameter problem has been considered, include bipartite graphs and graphs embeddable in a fixed surface (most notably, planar graphs).

### 2.4.1 Vertex-transitive graphs

Let $vt_{\Delta,D}$ be the largest order of a vertex-transitive $(\Delta, D)$-graph. As an aside, note that if a Moore graph of degree 57 and diameter 2 does exist then it cannot be vertex-transitive [72]. Somewhat surprisingly, although vertex-transitivity is a rather restrictive property, there is no better general upper bound on $vt_{\Delta,D}$ than the bounds listed in the previous sections. As regards lower bounds, a number of the existing examples of large $(\Delta, D)$-graphs are vertex-transitive (many of them actually are Cayley graphs and will be discussed in the next subsection).

The best general result here seems to be the one of McKay, Miller and Širáň [179] who showed that

$$vt_{\Delta,2} \geq \frac{8}{9}\Big(\Delta + \frac{1}{2}\Big)^2 \tag{5}$$

for all degrees of the form $\Delta = (3q-1)/2$, where $q$ is a prime power congruent to 1 (mod 4). The graphs that prove the inequality (5) are quite remarkable: They are all vertex-transitive but non-Cayley; the graph corresponding to the value $q = 5$ turns out

| $D$ / $\Delta$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 10 | 20 | 38 | 70 | 132 | 190 | 330 | 570 | 950 |
| 4 | 15 | 41 | 96 | 364 | 740 | 1 200 | 3 080 | 7 550 | 17 604 |
| 5 | 24 | 72 | 210 | 620 | 2 776 | 5 500 | 16 956 | 53 020 | 164 700 |
| 6 | 32 | 110 | 380 | 1 395 | 7 908 | 19 279 | 74 800 | 294 679 | 1 211 971 |
| 7 | 50 | 156 | 672 | 2 756 | 11 220 | 52 404 | 233 664 | 1 085 580 | 5 311 566 |
| 8 | 57 | 253 | 1081 | 5 050 | 39 672 | 129 473 | 713 539 | 4 039 649 | 13 964 808 |
| 9 | 74 | 585 | 1 536 | 7 884 | 75 828 | 270 048 | 1 485 466 | 8 911 766 | 25 006 478 |
| 10 | 91 | 650 | 2 211 | 12 788 | 134 690 | 561 949 | 4 019 489 | 13 964 808 | 52 029 411 |
| 11 | 98 | 715 | 3 200 | 18 632 | 156 864 | 970 410 | 5 211 606 | 48 626 760 | 179 755 200 |
| 12 | 133 | 780 | 4 680 | 29 435 | 359 646 | 1 900 319 | 10 007 820 | 97 386 380 | 466 338 600 |
| 13 | 162 | 845 | 6 560 | 39 402 | 531 440 | 2 901 294 | 15 733 122 | 145 880 280 | 762 616 400 |
| 14 | 183 | 912 | 8 200 | 56 325 | 816 186 | 6 200 460 | 34 839 506 | 194 639 900 | 1 865 452 680 |
| 15 | 186 | 1 215 | 11 712 | 73 984 | 1 417 248 | 7 100 796 | 45 000 618 | 282 740 976 | 3 630 989 376 |
| 16 | 198 | 1 600 | 14 640 | 132 496 | 1 771 560 | 14 882 658 | 86 882 544 | 585 652 704 | 7 394 669 856 |

Table 2: The order of the largest known graphs of maximum degree $\Delta$ and diameter $D$.

to be the Hoffman-Singleton graph, and for $q = 9$, the corresponding $(13, 2)$-graph has order 162, just 8 off the Moore bound $M_{13,2} = 170$.

The construction of McKay-Miller-Širáň graphs [179] relies on a suitable lift of the complete bipartite graph $K_{q,q}$. A simplified version (in the form of a lift of a dipole with $q$ edges and $(q-1)/4$ loops at each of its two vertices) was presented by Šiagiová [213], based on her results about compositions of regular coverings [212, 215]. In this connection it is interesting to mention another result of Šiagiová [214], who showed that, among all regular lifts of a dipole of degree $\Delta$, the maximum order of a lift of diameter 2 is, for sufficiently large $\Delta$, bounded above by

$$(4(10 + \sqrt{2})/49)\Delta^2 \simeq .93\Delta^2.$$

This compares well with the Moore bound $M_{\Delta,2} = \Delta^2 + 1$, and is larger than the bound from (5), which is approximately $.89\Delta^2$.

It is also worth noting that the graphs of McKay-Miller-Širáň are very rich in symmetries; their full automorphism groups were determined by Hafner [156], using ideas related to combinatorial geometry.

The results of [179] and [66] strongly suggest that computer search over lifts of small graphs, using various voltage assignments, may lead to further new examples of highly symmetric large graphs of given diameter and degree.

### 2.4.2 Cayley graphs

Let $C_{\Delta,D}$ denote the largest order of a Cayley graph of degree $\Delta$ and diameter $D$. The situation of general upper bounds for Cayley graphs of general groups is similar to the vertex-transitive case discussed above, with a few obvious exceptions. For instance, as the Petersen graph and the Hoffman-Singleton graph are known to be unique and non-Cayley, we have $C_{\Delta,D} \leq M_{\Delta,D} - 2$ for $D = 2$ and $\Delta = 3, 7$.

Lakshmivarahan, Jwo and Dhall [177] produced a survey of Cayley graph network designs. Apart from the usual properties of order, degree and diameter, they also consider shortest path distance, vertex-transitivity, arc-transitivity and several forms of distance transitivity. The survey emphasises algebraic features, such as cosets, conjugacy classes, and automorphism actions, in the determination of some topological properties of over 18 types of networks.

We note that roughly one half of the values in Table 2 have been obtained from Cayley graphs. Computer-assisted constructions of large $(\Delta, D)$-graphs, for relatively small $\Delta$ and $D$, from Cayley graphs of semidirect products of (mostly cyclic) groups can be found in Hafner [155]. Later, Branković *et al.* [66] showed that the constructions of [155] can be obtained as lifts of smaller Cayley graphs with voltage assignments in smaller, mostly cyclic, groups. Researchers who have contributed in the quest for large Cayley graphs of given degree and diameter also include Campbell [73], and Akers and Krishnamurthy [6].

An important stream of research in Cayley graphs, one that is closely related to the degree/diameter problem, is bounding the diameter of a Cayley graph in terms of a logarithm of the order of the group. The relation relies on the fact that, for $k \geq 3$ and $d \geq 2$, we have $M_{\Delta,D} < \Delta^D$, and therefore also $n < \Delta^D$ for $n = n_{\Delta,D}$. It follows that, for the diameter of a graph of order $n$, we always have

$$D > b \times \log n, \quad \text{where} \quad b = 1/\log \Delta.$$

Although taking logarithms results in a substantial loss of precision, it is still reasonable to ask about *upper* bounds on the diameter $D$ in terms of the logarithm of the largest order a $(\Delta, D)$-graph; as indicated earlier, this has been considered primarily for Cayley graphs.

From a result of Babai and Erdős [12], it follows that there exists a constant $c$, such that, for any finite group $G$, there exists a set of $t \leq c \log |G|$ generators, such that the associated Cayley graph has diameter at most $t$. This settles the general question about an upper bound on $D$, at least in terms of a constant multiple of the logarithm of $C_{\Delta,D}$, the largest order of a Cayley graph of degree $\Delta$ and diameter $D$. Further refinements have been obtained for special classes of groups, with emphasis on reducing the size of generating sets (and hence reducing the degree).

Babai, Kantor and Lubotzky [14] gave an elementary and constructive proof of the fact that every nonabelian finite simple group $G$ contains a set of at most *seven* generators for which the diameter of the associated Cayley graph is at most $c \log |G|$, for an absolute constant $c$. For projective special linear groups $G = PSL(m, q)$, this was improved by Kantor [169] by showing that, for each $m \geq 10$, there is a *trivalent* Cayley graph for $G$ of diameter at most $c \log |G|$.

For an *arbitrary* transitive subgroup $G$ of the symmetric group of degree $r$ and any symmetric generating set of $G$, Babai and Seress [16] proved that the diameter of the corresponding Cayley graph is at most

$$\exp((r \ln r)^{1/2}(1 + o(1))).$$

Note that this bound is quite far from

$$\log |G| = \log (r!) \approx cr \log r;$$

however, the strength of the statement is in that it is valid for arbitrary groups and generating sets. An earlier result by the same authors [15] states that if $G$ is either the symmetric or the alternating group of degree $r$, then, for an *arbitrary* symmetric generating set, the corresponding Cayley graph of $G$ has diameter not exceeding

$$\exp((r - \ln r)^{1/2}(1 + o(1))),$$

which is better than the previous bound (however, for more special groups). By probabilistic arguments, Babai and Hetyei [13] show that, for almost every pair of random permutations $(p_1, p_2)$ from the symmetric group of degree $r$, the diameter of the Cayley graph of the group $G = \langle p_1, p_2 \rangle$ with generating set $S = \{p_1^{\pm 1}, p_2^{\pm 1}\}$ is less than $\exp((\frac{1}{2} + o(1))(\ln r)^2)$. Since such a group almost surely (for $r \to \infty$) contains the alternating group of degree $r$, this result (at least in a probabilistic sense) is substantially stronger than the previous two bounds. Nevertheless, it is still far from the conjectured [15] upper bound $r^c$ for the diameter of *any* Cayley graph of the symmetric group of degree $r$, for an absolute constant $c$.

### 2.4.3 Abelian Cayley graphs

Further restrictions on the classes of groups yield better upper bounds. We discuss here in more detail the Cayley graphs of *abelian* groups. Let $AC_{\Delta,D}$ denote the largest order of a Cayley graph of an abelian group of degree $\Delta$ and diameter $D$. Inequalities for such graphs are often stated in terms of the number of generators of the reduced generating set rather than the degree. Given a Cayley graph $C(\Gamma, S)$, the reduced generating set is a subset $S'$ of $S$ such that, for each $s \in S$, exactly one of $s, s^{-1}$ appears in $S'$. If the reduced generating set has $d$ elements then the degree of the Cayley graph is equal to $\Delta = 2d - d'$, where $d'$ is the number of generators of order two in $S$.

Investigations of large abelian Cayley graphs of given size of reduced generating set and given diameter can be based on the following simple but ingenious idea (see [107] for genesis and background). Any finite abelian group $\Gamma$ with a symmetric generating set $S$ and a reduced generating set $S' = \{g_1, \ldots, g_d\}$ of size $d$ is a quotient group of the free abelian $d$-generator group $\mathcal{Z}^d$ by the subgroup $N$ (of finite index), that is, the kernel of the natural homomorphism $\mathcal{Z}^d \to \Gamma$, which sends the unit vector $\mathbf{e}_i \in \mathcal{Z}^d$ onto $g_i$. For any given $D$, define

$$W_{d,D} = \{(x_1, \ldots, x_d) \in \mathcal{Z}^d;\ |x_1| + \ldots + |x_d| \leq D\}.$$

Then the Cayley graph $C(\Gamma, S)$ has diameter at most $D$ if and only if $W_{d,D} + N = \mathcal{Z}^d$.

This has two immediate consequences. Firstly, $|W_{d,D}|$ is an upper bound on $AC_{2d,D}$. Secondly, if $N$ is a subgroup of $\mathcal{Z}^d$ of finite index with the property $W_{d,D} + N = \mathcal{Z}^d$ then $N$ determines a $d$-dimensional lattice that induces 'shifts of the set $W_{d,D}$ which completely cover the elements of $\mathcal{Z}^d$; the index $[\mathcal{Z}^d : N] = |\Gamma|$ (which is a lower bound on $AC_{2d,D}$) is also equal to the absolute value of the determinant of the $d$-dimensional matrix formed by the $d$ generating vectors of $N$. The search for bounds on $AC_{2d,D}$ can therefore be reduced to interesting problems in combinatorial geometry [107].

An exact formula for $|W_{d,D}|$ (which, as we know, is automatically an upper bound on $AC_{2d,D}$) was given, for example, by Stanton and Cowan [227]. A general lower bound on $AC_{2d,D}$, based on a thorough investigation of lattice coverings discussed above, was obtained by Dougherty and Faber [107]. We state both bounds in the following form: There exists a constant $c$ (not depending on $d$ and $D$), such that for any fixed $d \geq 2$ and all $D$,

$$\frac{c \times 2^d}{d!d(\ln d)^{1+\log_2 e}} D^d + O(D^{d-1}) \leq AC_{2d,D} \leq \sum_{i=0}^{d} 2^i \binom{d}{i} \binom{D}{i} \tag{6}$$

Note that the upper bound can be considered to be the *abelian Cayley Moore bound* for abelian groups with $d$-element reduced generating sets. It differs from the Moore bound $M_{2d,D}$ rather dramatically; if the number of generators $d$ is fixed and $D \to \infty$ then the right hand side of (6) has the form

$$2^d D^d / d! + O(D^{d-1}).$$

Exact values of $AC_{2d,D}$ are difficult to determine. With the help of lattice tilings, Dougherty and Faber (and many other authors, also using different methods – see [107]) showed that, for $d = 2$, there actually exist 'abelian Cayley Moore graphs', that is,

$$AC_{4,D} = |W_{2,D}| = 2D^2 + 2D + 1;$$

the analysis here is facilitated by a nice shape of the set $W_{2,D} \subset \mathcal{Z}^2$. For $d = 3$, the same type of analysis [107] gives

$$AC_{6,D} \geq (32D^3 + 48D^2)/27 + f(D),$$

where $f(D)$ is a linear function that depends on the residue class of $D$ mod 3; the abelian Cayley Moore bound here is

$$AC_{6,D} \leq |W_{3,D}| = (4D^3 + 6D^2 + 8D + 3)/3.$$

A table of exact values of $AC_{6,D}$ for $D \leq 14$ is included in [107].

It should be noted that the method of lattice-induced shifts of the sets $W_{d,D}$ tends to be manageable for small values of $d$ while $D \to \infty$, as can be seen from (6). At the other end of the spectrum, for diameter $D = 2$, a folklore result says that

$$AC_{\Delta,2} \geq \lfloor \frac{\Delta+2}{2} \rfloor \lceil \frac{\Delta+2}{2} \rceil \tag{7}$$

This can be obtained from a Cayley graph for the product of cyclic groups $Z_{\lfloor(\Delta+2)/2\rfloor} \times Z_{\lceil(\Delta+2)/2\rceil}$, with the generating set consisting of all pairs $(x_1, x_2)$, in which exactly one of $x_1, x_2$ is equal to 0. (The bound can in many cases be improved by 1 or 2.) Bounds on $AC_{\Delta,D}$, for small $D$ and $\Delta \to \infty$, can also be found in Garcia and Peyrat [137]; a typical result is that

$$AC_{\Delta,D} \geq \frac{\Delta^{D-2.17}}{21D!}$$

for $\Delta$ large enough and for $D \leq \Delta$.

### 2.4.4 Bipartite graphs

In this short subsection we consider Moore bound for bipartite graphs. The 'bipartite Moore bound', that is, the maximum number $B_{\Delta,D}$ of vertices in a bipartite graph of maximum degree $\Delta$ and diameter at most $D$, was given by Biggs [54]:

$$B_{2,D} = 2D \quad \text{and} \quad B_{\Delta,D} = \frac{2(\Delta-1)^D - 1}{\Delta - 2} \quad \text{if } \Delta > 2.$$

Bipartite graphs satisfying the equality are called *bipartite Moore graphs*. Apart from $K_2$, bipartite Moore graphs can exist only if $\Delta = 2$ (the $2\Delta$-cycles) or $D = 2$ (the complete bipartite graphs $K_{\Delta,\Delta}$), or if $D = 3, 4$ or 6 [54, 57]. For these values of $D$, bipartite Moore graphs exist if $\Delta - 1$ is a prime power [42, 54]. On the other hand, for $D = 3$, there are values of $\Delta$ with no bipartite Moore graphs. A study of semiregular bipartite Moore graphs was done by Yebra, Fiol and Fábrega [240].

Bond and Delorme [58, 59] give new constructions of large bipartite graphs with given degree and diameter, using their new concept of a partial Cayley graph. Other constructions of large bipartite graphs were found by Delorme [98, 99], using bipartite versions of operations described earlier, most notably, $*$-product and compounding. In the same papers, Delorme also studied the asymptotic behaviour of the problem by introducing the parameter

$$\beta_D = \lim\ \inf_{\Delta\to\infty} \frac{b_{\Delta,D}}{2\Delta^{D-1}}$$

where $b_{\Delta,D}$ is the largest order of a bipartite $(\Delta, D)$-graph. Comparing this with the bipartite Moore bound, we see that $\beta_D \leq 1$ for all $D$; so far, it is only known [98, 99] that equality holds for $D = 2, 3, 4$ and 6.

### 2.4.5 Graphs on surfaces

Let $\mathcal{S}$ be an arbitrary connected, closed surface (orientable or not) and let $n_{\Delta,D}(\mathcal{S})$ be the largest order of a graph of maximum degree at most $\Delta$ and diameter at most $k$, embeddable in $\mathcal{S}$. Let $\mathcal{S}_0$ be a sphere.

The planar (or, equivalently, spherical) version of the degree/diameter problem was considered by several authors. Hell and Seyffarth [159] have shown that, for diameter 2 and $\Delta \geq 8$, we have

$$n_{\Delta,2}(\mathcal{S}_0) = \lfloor \frac{3}{2}\Delta \rfloor + 1.$$

For $\Delta \leq 7$, the exact values of $n_{\Delta,2}(\mathcal{S}_0)$ were determined by Yang, Lin and Dai [239].

Subsequently, Fellows, Hell and Seyffarth [118] established upper and lower bounds for planar graphs of diameter 3 and any maximum degree $\Delta$ as

$$\lfloor \frac{9}{2}\Delta \rfloor - 3 \leq n_{\Delta,3}(\mathcal{S}_0) \leq 8\Delta + 12.$$

The case $\Delta = 3$ was also considered by Tishchenko [230]. For planar graphs with general diameter $D$ and with $\Delta \geq 4$, the authors in [118] (see also [119]) apply a special case of a theorem of Lipton and Tarjan [178], to show that

$$n_{\Delta,D}(\mathcal{S}_0) \leq (6D+3)(2\Delta^{\lfloor \frac{D}{2} \rfloor} + 1) \ .$$

An interesting generalisation of the result of [159] to arbitrary surfaces was obtained by Knor and Širáň [173]. Let $\mathcal{S}$ be an arbitrary surface (orientable or not) other than the sphere, and let $\Delta_{\mathcal{S}} = 2^8(2-\chi(\mathcal{S}))^2 + 2$. Then, for diameter $D = 2$ and any maximum degree $\Delta \geq \Delta_{\mathcal{S}}$,

$$n_{\Delta,2}(\mathcal{S}) = n_{\Delta,2}(\mathcal{S}_0) = \lfloor \frac{3}{2}\Delta \rfloor + 1.$$

The striking fact here is that this bound is, for $\Delta \geq \Delta_{\mathcal{S}}$, independent of the surface $\mathcal{S}$ and is the same as for the plane! The bound can therefore be considered to be the *surface Moore bound* for $\Delta \geq \Delta_{\mathcal{S}}$. In [173], it is also shown that, for all $\Delta \geq \Delta_{\mathcal{S}}$, there exist triangulations of $\mathcal{S}$ of diameter 2, maximum degree $\Delta$, and order $\lfloor (3/2)\Delta \rfloor + 1$; moreover, these 'surface Moore graphs' are not unique. The largest order of graphs of diameter 2 and degree at most 6 on surfaces with Euler characteristic $\geq 0$ was determined by Tishchenko [231].

Šiagiová and Simanjuntak [217] considered bounds on the order of graphs of arbitrary maximum degree $\Delta \geq 3$ and arbitrary diameter $D$, embeddable in a general surface of Euler genus $\varepsilon$. Setting $c_{\mathcal{S},D} = (6D(\varepsilon+1)+3)$, their result can be stated in the form

$$\frac{\Delta((\Delta-1)^{\lfloor \frac{D}{2} \rfloor} - 2)}{\Delta - 2} < n_{\Delta,D}(\mathcal{S}) \leq c_{\mathcal{S},D} \frac{\Delta((\Delta-1)^{\lfloor \frac{D}{2} \rfloor} - 2)}{\Delta - 2}.$$

In view of these bounds, the authors of [217] raise the natural question of the existence and the value of the limit of $n_{\Delta,D}(\mathcal{S})/\Delta^{\lfloor D/2 \rfloor}$ as $\Delta \to \infty$.

## 2.5 Related topics

The relationships between parameters such as order, diameter, minimum degree and maximum degree have been considered by Chung [84]. She reviews the status of a number of interrelated problems on diameters of graphs, including: (i) degree/diameter problem, (ii) order/degree problem, (iii) given $n, D, D', s$, determine the minimum number of edges in a graph on $n$ vertices of diameter $D$ having the property that after removing any $s$ or fewer edges the remaining graph has diameter at most $D'$, (iv) problem (iii) with a constraint on the maximum degree, (v) for a given graph, find the optimum way to add $t$ edges so that the resulting graph has minimum diameter, and (vi) for a given graph, find the optimum way to add $t$ vertex disjoint edges to reduce the diameter as much as possible.

A variety of interrelated diameter problems are discussed by Chung in [83], including determining extremal graphs of bounded degrees and small diameters, finding orientations for undirected or mixed graphs to minimise diameters, investigating diameter bounds for networks with possible node and link failures, and algorithmic aspects of determining the diameters of graphs.

In her study of properties of eigenvalues of the adjacency matrix of a graph, Chung [85] proved that the second largest eigenvalue (in absolute value) $\lambda$ is related to the diameter $D$ by means of the inequality

$$D \leq \lceil \log(n-1)/\log(\Delta/\lambda) \rceil.$$

Bermond and Bollobás [39] studied the following extremal problem: Given integers $n$, $D$, $D'$, $\Delta$, $k$ and $l$, determine or estimate the minimum number of edges in a graph $G$ of order $n$ and with the following properties: (i) $G$ has maximum degree at most $\Delta$, (ii) the diameter of $G$ is at most $D$, (iii) if $G'$ is obtained from $G$ by suppressing any $k$ of the vertices or any $l$ of the edges, the diameter of $G'$ is at most $D'$.

Bollobás [57] considered another extremal problem on diameters: given diameter and maximum degree, find the minimum number of edges.

Gómez and Escudero [145] investigated constructions of graphs with a given diameter $D$ and a given maximum degree $\Delta$ and having a large number of vertices, whose edges can be well coloured by exactly $p$ colours. They include a table of such digraphs for $D \leq 10$ and $p \leq 16$.

The two additional parameters that have been considered most systematically in relation with the degree/diameter problem are *girth* (= length of the shortest cycle) and *connectivity*; we consider them in separate subsections.

### 2.5.1 Girth

Biggs [55] studied the number of vertices of a regular graph whose girth and degree are given. If the degree is $D \geq 3$ and girth $g = 2r + 1$, $r \geq 2$, then there is a simple lower bound

$$n_0(g, D) = 1 + \frac{D}{D-2}((D-1)^r - 1)$$

for the number of vertices. It has been proved by Bannai and Ito [19], and by Damerell [95], that the bound can be attained only when $g = 5$ and $D = 3, 7$ or 57. For related work, see also Biggs and Ito [56].

On the other hand, attempts to find general constructions for graphs with given girth and degree have yielded only much larger graphs than the lower bound. Bollobás [57] gives an overview of the problem and presents open questions regarding the behaviour of the number of excess vertices $n - n_0(g, D)$, where $n$ is the smallest possible order. Cubic (that is, trivalent) graphs of a given girth and with the smallest possible number of vertices have been known as *cages*. For a survey article about cages, we recommend Wong [236]; for latest results, the interested reader should consult Exoo [113].

Using matrix theory, Bannai and Itoh [20] proved that there do not exist any regular graphs with excess 1 and girth $2r + 1 \geq 5$, and that, for $r \geq 3$, there are no antipodal regular graphs with diameter $r + 1$ and girth $2r + 1$.

Dutton and Brigham [108] gave upper bounds for the maximum number of edges $e$ possible in a graph depending upon its order $n$, girth $g$ (and sometimes minimum degree $\delta$).

### 2.5.2 Connectivity

Chung, Delorme and Solé [86] define the $k$-diameter of a graph $G$ as the largest pairwise minimum distance of a set of $k$ vertices in $G$, i.e., the best possible distance of a code of size $k$ in $G$. They study a function $N(k, \Delta, D)$, the largest size of a graph of degree at most $\Delta$ and $k$-diameter $D$, and give constructions of large graphs with given degree and $k$-diameter. They also give upper bounds for the eigenvalues, and new lower bounds on spectral multiplicity.

A parameter which is believed to be particularly important in networks is the reliability of the network: it is desirable that if some stations (resp., branches) are unable to work, the message can still be always transmitted. This corresponds to the connectivity (resp., edge-connectivity) of the associated graph. It is well known that the connectivity is less than or equal to the edge-connectivity, which is less than or equal to the minimum degree of the graph.

Seidman [210] gives an upper bound for the diameter of a connected graph in terms of its number of vertices, minimum degree and connectivity. Earlier results in this direction were also obtained by Watkins [233] and Kramer [175].

Bauer, Boesch, Suffel and Tindell [36] introduced the notion of super-$\lambda$ graphs for the study of network reliability. A graph is *super*-$\lambda$ if every edge cut of minimum size is an edge cut isolating a vertex. Soneoka [225] surveyed sufficient conditions for connectivity or edge-connectivity to be equal to the minimum degree. Additionally, the author proved a sufficient condition for super-$\lambda$ in terms of the diameter $D$, order $n$, minimum degree $\delta$ and maximum degree $\Delta$. He proves that a graph is super-$\lambda$ if

$$n \geq \delta(((\Delta - 1)^{D-1} - 1)/(\Delta - 2) + 1) + (\Delta - 1)^{D-1}.$$

The bounds are best possible for graphs with diameter 2,3,4 and 6.

Fiol [120] considers the relation between connectivity (resp., superconnectivity) and other parameters of a graph $G$, namely, its order $n$, minimum degree, maximum degree, diameter, and girth.

Using the same parameters, Balbuena, Carmona, Fábrega and Fiol [17] show that the connectivity, as well as arc-connectivity, of a bipartite graph is maximum possible, provided that $n$ is large enough.

Quaife [204] gives an overview and some new results concerning the optimisation problem of the order of a graph given maximum degree, diameter and another parameter $\mu$ which expresses a redundancy. An undirected finite graph $G$ is a $(\Delta, D, \mu)$-graph if, for each pair of distinct vertices of $G$, there exist at least $\mu$ edge-disjoint paths joining these vertices, each path consisting of $k$ or fewer edges. The original $(\Delta, D)$ problem is then the $(\Delta, D, 1)$ problem.

Other papers relating the order of a graph, its maximum degree and diameter (and possibly other parameters) with the connectivity of a graph, include the studies by Fiol [120, 121], Fiol, Fábrega and Escudero [123], Bermond, Homobono and Peyrat [51], [52].

# 3 Part 2: Directed graphs

## 3.1 Moore digraphs

As in the case of undirected graphs, there is a natural upper bound $n_{d,k}$ on the order of directed graphs (digraphs), given maximum out-degree $d$ and diameter $k$. For any given vertex $v$ of a digraph $G$, we can count the number of vertices at a particular distance from that vertex. Let $n_i$, for $0 \leq i \leq k$, be the number of vertices at distance $i$ from $v$. Then $n_i \leq d^i$, for $0 \leq i \leq k$, and consequently,

$$\begin{aligned} n_{d,k} = \sum_{i=0}^{k} n_i &\leq 1 + d + d^2 + \ldots + d^k \\ &= \begin{cases} \frac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1 \end{cases} \end{aligned} \tag{8}$$

The right-hand side of (8), denoted by $M_{d,k}$, is called the *Moore bound* for digraphs. If the equality sign holds in (8) then the digraph is called a *Moore digraph.*

It is well known that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k+1$, $C_{k+1}$, for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d+1$, $K_{d+1}$, for any $d \geq 1$). This was first proved by Plesník and Znám in 1974 [203] and later independently by Bridges and Toueg who presented in 1980 a short and very elegant proof [67].

Throughout, a digraph of maximum out-degree $d$ and diameter $k$ will be referred to as *$(d,k)$-digraph.* Since there are no Moore $(d,k)$-digraphs for $d \geq 2$ and $k \geq 2$, the study of the existence of large digraphs focuses on $(d,k)$-digraphs whose order is close to the Moore bound, that is, digraphs of order $n = M_{d,k} - \delta$, where the *defect* $\delta$ is as small as possible.

## 3.2 Digraphs of order close to Moore bound

We start this section with a survey of the existence of digraphs of order one less than the Moore bound, that is, with $(d,k)$-digraphs of defect one; such digraphs are alternatively called *almost Moore digraphs.*

For the diameter $k = 2$, line digraphs of complete digraphs are examples of almost Moore digraphs for any $d \geq 2$, showing that $n_{d,2} = M_{d,2} - 1$. Interestingly, for out-degree $d = 2$, there are exactly three non-isomorphic diregular digraphs of order $M_{2,2} - 1$: the line digraph of $K_3$ plus two other digraphs (cf [184]), see Fig. 5. However, for maximum out-degree $d \geq 3$, Gimbert [138, 140] completely settled the classification problem for diameter 2 when he proved that line digraphs of complete digraphs are the only almost Moore digraphs.

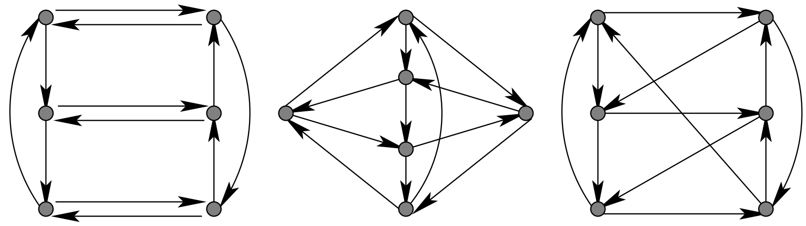

Figure 5: Three non-isomorphic diregular digraphs of order $M_{2,2} - 1$.

On the other hand, focusing on small out-degree instead of diameter, Miller and Fris [184] proved that there are no almost Moore digraphs of maximum out-degree 2, for any $k \geq 3$. Moreover, a recent result of Baskoro, Miller, Širáň and Sutton [34] shows that there are no almost Moore digraphs of maximum out-degree 3 and any diameter greater than or equal to 3. The question of whether or not the equality can hold in $n_{d,k} \leq M_{d,k} - 1$, for $d \geq 4$ and $k \geq 3$, is completely open.

The study of digraphs of defect 2 has so far concentrated on digraphs of maximum out-degree $d = 2$. In the case of diameter $k = 2$, it was shown by Miller [182] that there are exactly five non-isomorphic diregular digraphs of defect 2. In [182], Miller proved the non-existence of digraphs of defect two for out-degree 2 and diameter $k \geq 3$ by deriving a necessary condition, namely, that $k+1$ must divide $2(2^{k+1}-3)$, the number of arcs in the digraph of defect 2. Interestingly, this condition excludes many values of $k$. For example, for $3 \leq k \leq 10^7$ there are only two values ($k = 274485$ and $k = 5035921$) for which the divisibility condition holds. Consequently, for all but these two values of $k$, $3 \leq k \leq 10^7$, it has been known for some time that digraphs of defect 2 do not exist for out-degree $d = 2$. Miller and Širáň [187] improved this result by showing that digraphs of defect 2 do not exist for out-degree $d = 2$ and all $k \geq 3$.

For the remaining values of $k \geq 3$ and $d \geq 3$, the question of whether digraphs of defect 2 exist or not remains completely open; for examples of recent work, see Miller *et al.* [185, 187]. Our current knowledge of the upper bound on the order of digraphs of out-degree $d$ and diameter $k$ is summarised in Table 3.

A number of structural and non-existence results concerning almost Moore digraphs (the $(d, k)$-digraphs of defect one) are based on the following concept. If $G$ is an almost Moore digraph then for each vertex $v \in V(G)$ there exists exactly one vertex, denoted by $r(v)$ and called the *repeat* of $v$, such that there are exactly two $v \to r(v)$ walks of length at most $k$. If $S$ is a set (resp. multiset) of vertices then $r(S)$ is the set or a multiset of all the repeats of all the elements of $S$. We denote by $N^+(u)$ the set (or multiset) of the out-neighbours of a vertex $u$, and we denote by $N^-(u)$ the set (or multiset) of the in-neighbours of $u$.

If the almost Moore digraph $G$ is diregular then the map $r$ that assigns to each vertex $v \in V(G)$ its repeat $r(v)$ is an automorphism of $G$. This follows from the *Neighbourhood*

| *Diameter k* | *Degree d* | *Upper Bound of order* $n_{d,k}$ |
|---|---|---|
| $k = 1$ | all $d \geq 1$ | $M_{d,1}$ |
| $k = 2$ | $d = 1$ | $M_{1,2}$ |
| | all $d \geq 2$ | $M_{d,2} - 1$ |
| $k \geq 3$ | $d = 1$ | $M_{1,k}$ |
| | $d = 2$ | $M_{2,k} - 3$ |
| | $d = 3$ | $M_{3,k} - 2$ |
| | all $d \geq 4$ | $M_{d,k} - 1$ |

Table 3: Upper bounds on the order of digraphs of degree $d$ and diameter $k$.

*Lemma* of Baskoro, Miller, Plesník and Znám [31] which asserts that $N^+(r(v)) = r(N^+(v))$ and $N^-(r(v)) = r(N^-(v))$ for any vertex $v$ of a diregular almost Moore digraph. Moreover, the permutation matrix $P$ associated with the automorphism $r$ (viewed as a permutation on the vertex set of the digraph) satisfies the equation

$$I + A + A^2 + \ldots + A^k = J + P,$$

where $A$ is the adjacency matrix of $G$ and $J$ denotes the $n \times n$ matrix of all 1's.

The rest of the results mentioned in this section have been proved with the help of repeats (often combined with other techniques, most notably, matrix methods).

Miller and Fris [184] proved that there are no almost Moore digraphs for $d = 2$ and $k \geq 3$.

Baskoro, Miller, Plesník and Znám [31] gave a necessary divisibility condition for the existence of (diregular) almost Moore digraphs of degree 3, namely that if a diregular almost Moore digraph of degree 3 and diameter $k \geq 3$ exists then $k+1$ divides $\frac{9}{2}(3^k - 1)$. Using this condition they deduce that such digraphs do not exist for infinitely many values of the diameter (if $k$ is odd or if 27 divides $k + 1$).

Baskoro, Miller, Plesník and Znám [30] considered diregular almost Moore digraphs of diameter 2. Using the eigenvalues of adjacency matrices, they give several necessary conditions for the existence of such digraphs. For degree 3, they prove that there is no such digraph other than the line digraph of the complete digraph $K_4$ (a Kautz digraph).

For diregular digraphs, Baskoro, Miller and Plesník [32] gave various properties of repeats and structural results involving repeats and especially selfrepeats (vertices for which

$r(v) = v$. These culminate in the theorem stating that for $d \geq 3$, $k \geq 3$, an almost Moore digraph contains either no selfrepeats or exactly $k$ selfrepeats, that is, an almost Moore digraph contains at most one $C_k$.

In [33] Baskoro, Miller and Plesník gave further necessary conditions for the existence of almost Moore digraphs. They consider the cycle structure of the permutation $r$ (repeat) and find that certain induced subdigraphs in a diregular almost Moore digraph are either cycles or, more interestingly, smaller almost Moore digraphs. For $k = 2$ and degree $2 \leq d \leq 12$ they show that if there is a $C_2$ then every vertex lies on a $C_2$ (that is, all vertices are selfrepeats or none is).

Baskoro, Miller and Širáň [35] studied almost Moore digraphs of degree 3 and found that such a digraph cannot be a Cayley digraph of an abelian group.

Gimbert [139] dealt with the problem of $(h, k)$-digraphs, where there is a unique directed walk of length at least $h$ and at most $k$ between any two vertices of the digraph and found that such digraphs exist only when $h = k$ and $h = k - 1$ if $d \geq 2$. In the cases of $d = 2$ or $k = 2$, it is shown, using algebraic techniques, that the line digraph $L(K_{d+1})$ of the complete digraph $K_{d+1}$ is the only $(1, 2)$-digraph of degree $d$, that is, the only digraph whose adjacency matrix $A$ satisfies the equation $A + A^2 = J$. As a consequence, there does not exist any other almost Moore digraph of diameter $k = 2$ with all selfrepeat vertices apart from the Kautz digraph.

Gimbert [138] used the characteristic polynomial of an almost Moore digraph to obtain some new necessary conditions for the cycle structure of the automorphism $r$ of such a digraph. In particular, he applied the results to the cases of diameters 2 and 3 and proved that there is exactly one almost Moore digraph for $d = 4$ and $k = 2$, the line digraph of $K_5$.

Inspired by the technique of Bridges and Toueg [67], Baskoro, Miller, Plesník and Znám [31] used matrix theory (the eigenvalues of the adjacency matrix) to prove that there is no diregular almost Moore digraph of degree $\geq 2$, diameter $k \geq 3$ and with every vertex a selfrepeat, that is, every vertex on a directed cycle $C_k$. Note that Bosák [61] already studied diregular digraphs satisfying the more general matrix equation

$$A^a + A^{a+1} + \ldots + A^b = J, \; a \leq b,$$

and he proved that for $d > 1$ such digraphs exist only if either $b = a$ (de Bruijn digraphs [69]) or $b = a + 1$ (Kautz digraphs [170, 171]). Thus the result of [31] is only the case $a = 1$ and $b = k \geq 3$ but we mention it here because the proof is much simpler than Bosák's proof.

Cholily, Baskoro and Uttunggadewa [81] gave some conditions for the existence of almost Moore digraphs containing selfrepeat. The smallest positive integer $p$ such that the composition $r^p(u) = u$ is called the *order* of $u$. Baskoro, Cholily and Miller [26, 27] investigated the number of vertex orders present in an almost Moore digraphs containing selfrepeat. An exact formula for the number of all vertex orders in a graph is given, based on the vertex orders of the outneighbours of any selfrepeat vertex.

### 3.3 Diregularity of digraphs close to Moore bound

We shall next consider the question of diregularity of digraphs, given maximum out-degree $d$ and diameter $k$. To get a more complete picture, we make a short detour and briefly consider the much simpler issue of the regularity of undirected graphs.

For undirected graphs, if there is a vertex of degree less than $\Delta$ then the order of the graph cannot be more than

$$\begin{aligned} n_{\Delta,D} = \sum_{i=0}^{D} n_i &\leq 1 + (\Delta-1) + (\Delta-1)(\Delta-1) + \ldots + (\Delta-1)(\Delta-1)^{D-1} \\ &= 1 + (\Delta-1)(1 + (\Delta-1) + \ldots + (\Delta-1)^{D-1}) \\ &= \begin{cases} 1 + (\Delta-1)\,\frac{(\Delta-1)^D-1}{\Delta-2} = M_{\Delta,D} - \frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta > 2 \\ D+1 = M_{2,D} - D & \text{if } \Delta = 2 \end{cases} \end{aligned} \qquad (9)$$

Obviously, it follows that graphs with the number of vertices 'close' to the Moore bound cannot have any vertex of degree less than $\Delta$, that is, the graphs are necessarily regular, end of story. However, for directed graphs the situation is much more interesting.

The only strongly connected digraph of out-degree $d = 1$ is the directed cycle $C_{k+1}$. For $d > 1$, if the maximum out-degree is $d$ and if there is a vertex of out-degree less than $d$ then we have

$$\begin{aligned} n_{d,k} = \sum_{i=0}^{k} n_i &\leq 1 + (d-1) + (d-1)d + \ldots + (d-1)d^{k-1} \\ &= (1 + d + d^2 + \ldots + d^k) - (1 + d + d^2 + \ldots + d^{k-1}) \\ &= M_{d,k} - M_{d,k-1}. \end{aligned} \qquad (10)$$

Therefore, a digraph of maximum out-degree $d \geq 2$, diameter $k$ and order $n = M_{d,k} - \delta$ must be out-regular if $\delta < M_{d,k-1}$. However, establishing the regularity or otherwise of the in-degree of digraphs (given maximum out-degree) is not so straightforward. Indeed, there exist digraphs of out-degree $d$ and diameter $k$, whose order is just two or three less than the Moore bound and in which not all vertices have the same in-degree. These graphs are out-regular but not in-regular.

For example, when $d = 2$, $k = 2$, $n = 5$ (that is, defect 2), there are 9 non-isomorphic digraphs. Of these, 5 are diregular (see Fig. 6) and 4 are non-diregular (see Fig. 7).

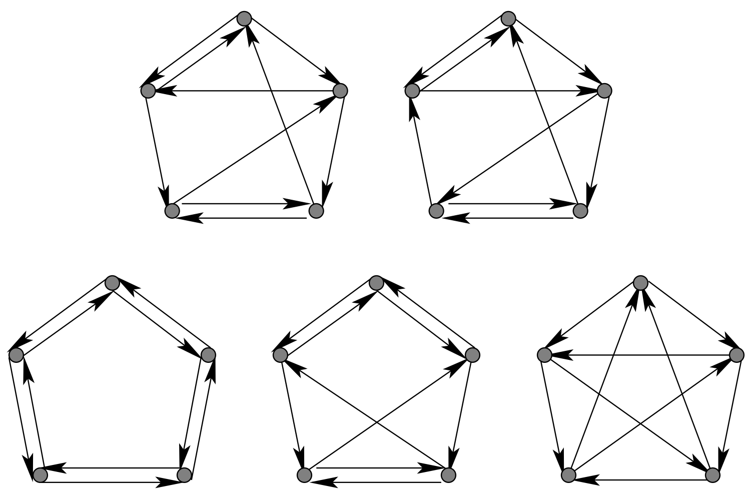

Figure 6: Five non-isomorphic diregular digraphs of order $M_{2,2} - 2$.

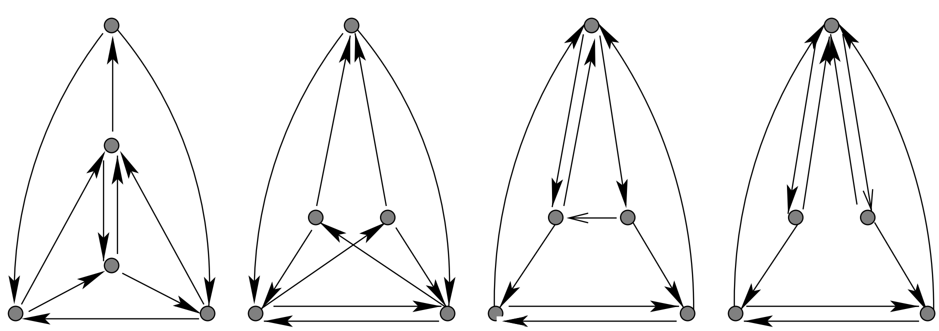

Figure 7: Four non-isomorphic non-diregular digraphs of order $M_{2,2} - 2$.

It is interesting to note that there are more diregular digraphs than non-diregular ones for the parameters $n = 5$, $d = 2$, $k = 2$, while for the next larger digraphs of defect 2, namely, when $n = 11$, $d = 3$, $k = 2$, the situation is quite different: there are at least four non-isomorphic non-diregular digraphs [220] but only one diregular digraph [23] (see Figs. 8 and 9.

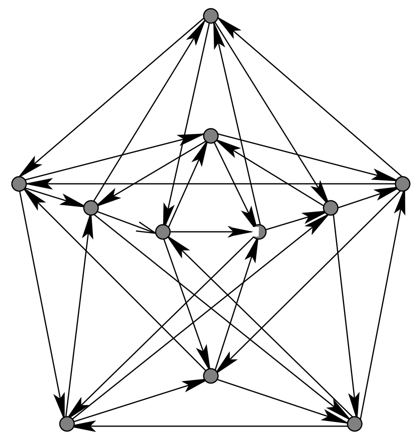

Figure 8: The unique diregular digraph of order $M_{3,2} - 2$.

Miller, Gimbert, Širáň and Slamin [185] proved that every almost Moore digraph is diregular. Miller and Slamin [188] proved that every digraph of defect 2, maximum out-degree 2 and diameter $k \geq 3$ is diregular. Slamin, Baskoro and Miller [221] studied diregularity of digraphs of defect 2 and maximum out-degree 3. Miller and Slamin conjecture that all defect 2 digraphs of maximum out-degree $d \geq 2$ are diregular, provided $k \geq 3$.

The question of diregularity or otherwise of digraphs with defect greater than 2 is completely open.

## 3.4 Constructions of large digraphs

The best lower bound on the order of digraphs of maximum out-degree $d$ and diameter $k$ is as follows. For maximum out-degree $d \geq 2$ and diameter $k \geq 4$,

$$n_{d,k} \geq 25 \times 2^{k-4}. \qquad (11)$$

This lower bound is obtained from the *Alegre digraph* [7] which is a diregular digraph of degree 2, diameter 4 and order 25 (see Fig. 10), and from its iterated line digraphs. For the remaining values of maximum out-degree and diameter, a general lower bound is

$$n_{d,k} \geq d^k + d^{k-1}. \qquad (12)$$

This bound is obtained from *Kautz digraphs*, that is, the diregular digraphs of degree $d$,

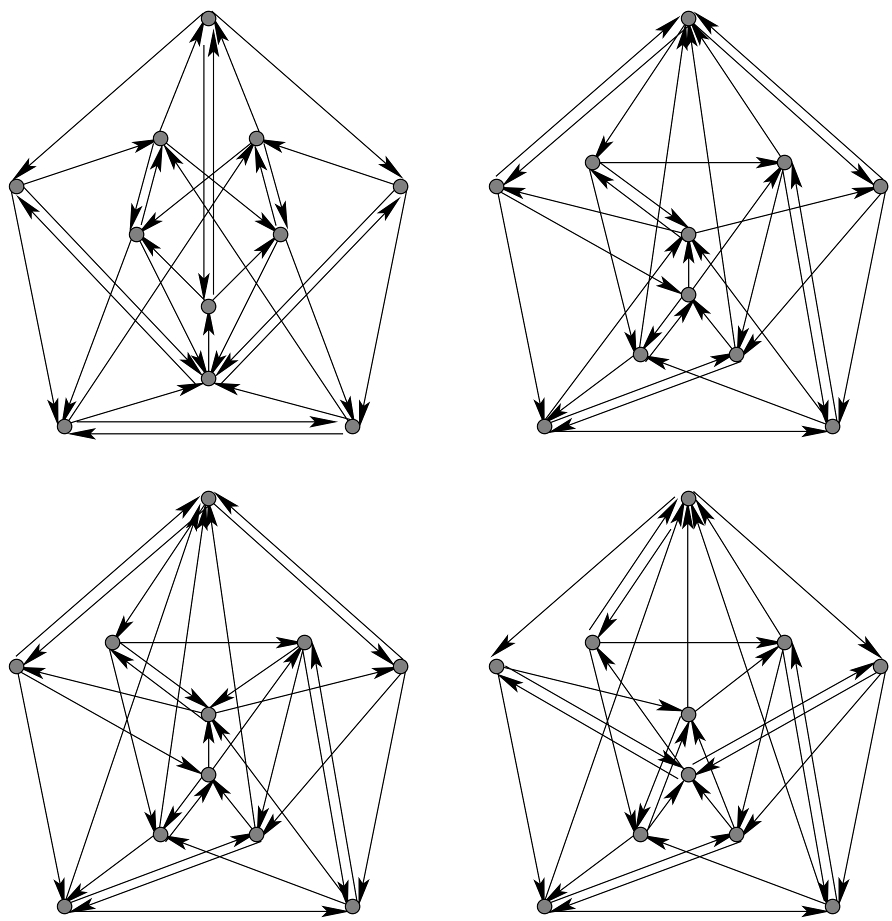

Figure 9: Four non-isomorphic non-diregular digraphs of order $M_{3,2} - 2$.

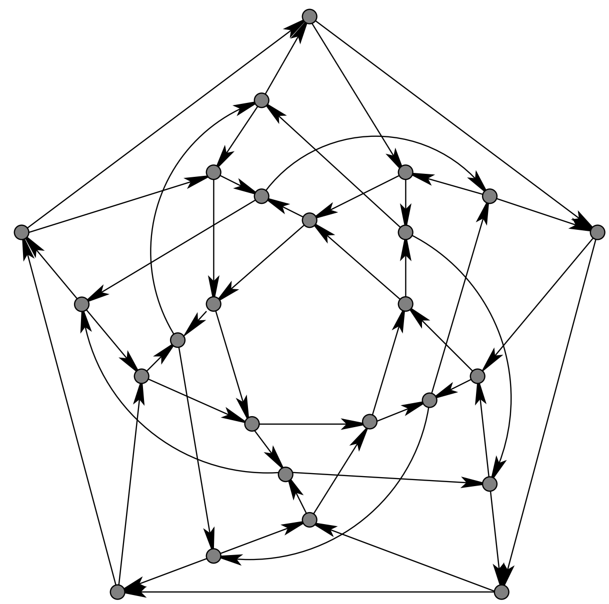

Figure 10: Alegre digraph.

diameter $k$ and order $d^k + d^{k-1}$ [170]. Kautz digraphs, although defined in the literature in various ways, are just iterated line digraphs of complete digraphs (as an example, see the Kautz digraph on 24 vertices of degree 2 and diameter 4, in Fig. 11). Line digraph iterations were also studied by Fiol, Alegre and Yebra [122]; for a nice partial line digraph technique, see Fiol and Lladó [129]. For an example of recent work related to Kautz and de Bruijn digraphs, we refer to Barth and Heydemann [21].

In [163, 164], Imase and Itoh considered the minimum diameter problem and the lower bound for diameter $k$, given the number of nodes $n$ and the in- and out-degree of each node being $d$ or less. From the Moore bound, they obtained

$$k \geq \lceil \log_d(n(d-1)+1) \rceil - 1 = l(n,d),$$

where $1 < d \leq n-1$ and $\lceil x \rceil$ denotes the minimum integer not smaller than $x$. In [163], they gave the following construction of a $(d,k)$-digraph of order $n$ with diameter $k = \lceil \log_d n \rceil$. For any $n$ and $d$ $(1 < d \leq n-1)$, the vertex set of the digraph is $\{0, 1, \ldots, n-1\}$, and there is an arc from $i$ to $j$ if and only if $j \equiv id + q \pmod{n}$, $q = 0, 1, \ldots, d-1$. The diameter of the digraph is either equal to $l(n,d)$ or is at most one more. If $\frac{d^k-1}{d-1} \leq n \leq d^k$ or if $n = d^{k-b}(d^b+1)$, $b$ odd, $b \leq m$, then the construction achieves diameter equal to the lower bound $l(n,d)$.

The construction by Imase and Itoh [164] was improved by Baskoro and Miller [28] who produced a construction for digraphs of $d^{k-b}(d^b+1)$ vertices for any $b$ (note that the construction in [164] worked only for $b$ odd). The procedure makes use of de Bruijn digraphs [69]. For other constructions based on adjacency defined by congruence relations, we refer to Opatrný [199, 200], and to Gómez, Padró and Perennes [151].

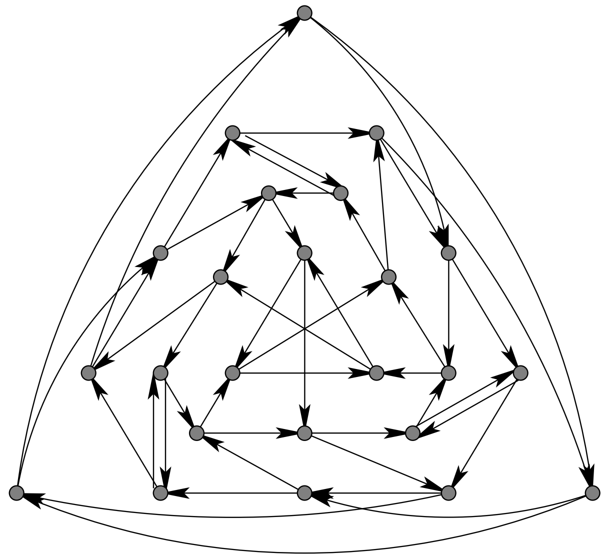

Figure 11: Kautz digraph.

Examples of large digraphs of given degree and diameter have also been constructed by heuristic search, see e.g., Allwright [9].

The lifting method described in Subsection 2.3.2 is suitable for constructing digraphs as well as undirected graphs. In fact, most of the concepts introduced in Subsection 2.3.2 apply to digraphs with no or just minor changes. Let $G$ be a *base digraph* with arc set $A(G)$ and let $\Gamma$ be a finite group. This time, a *voltage assignment* on $G$ in $\Gamma$ is *any* mapping $\alpha: A(G) \to \Gamma$; no extra condition on voltages is needed because edge directions are a part of the description of the digraph $G$. The definition of the lift $G^\alpha$ is formally the same as in Subsection 2.3.2, and the lift is automatically a digraph.

For an example, we refer to Fig. 12 that shows how the Alegre digraph can be obtained as a lift of a base digraph of order 5, endowed with voltages in the group $\mathcal{Z}_5$.

Lifts of graphs implicitly appear in a number of constructions of large $(d, k)$-digraphs. To our knowledge, the first to explicitly use lifts of graphs were Annexstein, Baumslag and Rosenberg [10] in connection with their group action graphs. Such graphs were then later studied by Espona and Serra [110] to produce large Cayley $(d, k)$-digraphs based on the so-called de Bruijn networks. We recall that, given a group $\Gamma$ and an arbitrary generating sequence $Y$ of elements $y_1, y_2, \ldots, y_d$ of $\Gamma$, the Cayley digraph $C(\Gamma, Y)$ has vertex set $\Gamma$, and for each $g \in \Gamma$ and each $y_i \in Y$, there is a directed edge from $g$ to $gy_i$.

We point out that the role of lifts in the context of digraphs is similar to the situation we have encountered in undirected graphs, and the reasons are essentially the same. To name a few of the advantages of lifts, the diameter of the lifted digraph can be expressed in terms

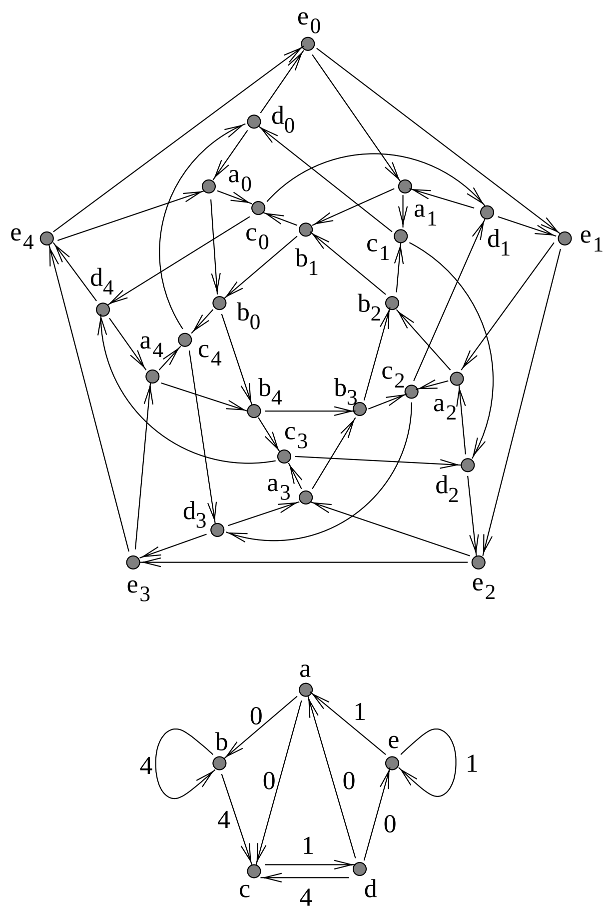


Figure 12: A base graph $G$ with voltage assignment in $\mathcal{Z}_5$ and its lift, the Alegre digraph.

of voltages on walks of the base digraph [25], which can be used to design efficient diameter-checking algorithms. Further, if a digraph contains a non-trivial group of automorphisms acting freely on its vertex set, then the digraph is a lift of a smaller digraph. This remark, which directly follows from [153], applies to most currently known largest examples of $(d, k)$-digraphs, and so most of them can be described as lifts. Likewise, all constructions where incidence is defined by linear congruences are, in fact, lifting constructions.

Any Cayley digraph $C(\Gamma, S)$ is a lift of a single-vertex digraph (with $|S|$ directed loops carrying voltages from the generating set $S$). Additionally, quite complex Cayley digraphs that have appeared in the directed version of the degree/diameter problem (such as the ones of certain semidirect products of Abelian groups considered in [105]) can be described as ordinary lifts of smaller Cayley digraphs, with voltages in Abelian (mostly cyclic) groups; see [66]. As regards transitivity, convenient sufficient conditions can be extracted from [66] for a lift to be a vertex transitive (or a Cayley) digraph, which is suitable for producing large vertex transitive $(d, k)$-digraphs by lifts.

A theoretical background for lifts in the study of large $(d, k)$-digraphs can be found in Baskoro *et al.* [25] and Branković *et al.* [66]. Some of the results of [25] were in particular cases strengthened by Zlatoš [242], who proved several upper bounds on the diameter of the lift in terms of some properties of the base digraph and the voltage group. A number of his results give significantly improved upper bounds when the digraph is a Cayley digraph and the voltage group is abelian.

Table 4 gives a summary of the current largest known digraphs for maximum out-degree $d \leq 13$ and diameter $k \leq 11$.

## 3.5 Restricted versions of the degree/diameter problem

Unlike the undirected case, the restrictions of the degree/diameter problem for digraphs that have been considered in the literature are mostly connected with vertex transitivity. Issues such as biparticity and connectivity have received less attention so far.

### 3.5.1 Vertex-transitive digraphs

Let $vt_{d,k}$ be the largest order of a vertex-transitive digraph of maximum out-degree $d$ and diameter $k$. Obviously, we have $vt_{d,k} = M_{d,k}$ if $d = 1$ or if $k = 1$. Moreover, as line digraphs of complete digraphs are vertex-transitive, we also have $vt_{d,2} = M_{d,2} - 1$, for all $d \geq 2$. Apart from this, there do not seem to be any general upper bounds on $vt_{d,k}$. Constructions that yield lower bounds on $vt_{d,k}$ rely mostly on coset graphs or on certain compositions.

Let $\Gamma$ be a finite group, let $\Lambda$ be a subgroup of $\Gamma$, and let $X$ be a set of distinct $\Lambda$-coset representatives, such that $\Gamma$ is generated by $\Lambda \cup X$, $X \cap \Lambda = \emptyset$, and $\Lambda X \Lambda \subseteq X\Lambda$. The

| d \ k | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | **6** | **12** | **25** | **50** | **100** | **200** | **400** | **800** | **1 600** | **3 200** |
| 3 | **12** | **36** | **108** | **324** | **972** | **2 916** | **8 748** | **26 244** | **78 732** | **236 196** |
| 4 | **20** | **80** | **320** | **1 280** | **5 120** | **20 480** | **81 920** | **327 680** | **1 310 720** | **5 242 880** |
| 5 | **30** | **150** | **750** | **3 750** | **18 750** | **93 750** | **468 750** | **2 343 750** | **11 718 750** | **58 593 750** |
| 6 | **42** | **252** | **1 512** | **9 072** | **54 432** | **326 592** | **1 959 552** | **11 757 312** | **70 543 872** | **423 263 232** |
| 7 | **56** | **392** | **2 744** | **19 208** | **134 456** | **941 192** | **6 588 344** | **46 118 408** | **322 828 856** | **2 259 801 992** |
| 8 | **72** | **576** | **4 608** | **36 864** | **294 912** | **2 359 296** | **18 874 368** | **150 994 944** | **1 207 959 552** | **9 663 676 416** |
| 9 | **90** | **810** | **7 290** | **65 610** | **590 490** | **5 314 410** | **47 829 690** | **430 467 210** | **3 874 204 890** | **34 867 844 010** |
| 10 | **110** | **1 100** | **11 000** | **110 000** | **1 100 000** | **11 000 000** | **110 000 000** | **1 100 000 000** | **11 000 000 000** | **110 000 000 000** |
| 11 | **132** | **1 452** | **15 972** | **175 692** | **1 932 612** | **21 258 732** | **233 846 052** | **2 572 306 572** | **28 295 372 292** | **311 249 095 212** |
| 12 | **156** | **1 872** | **22 464** | **269 568** | **3 234 816** | **38 817 792** | **465 813 504** | **5 589 762 048** | **67 077 144 576** | **804 925 734 912** |
| 13 | **182** | **2 366** | **30 758** | **399 854** | **5 198 102** | **67 575 326** | **878 479 238** | **11 420 230 094** | **148 462 991 222** | **1 930 018 885 886** |

Table 4: The order of the largest known digraphs of maximum out-degree $d$ and diameter $k$.

*Cayley coset digraph* $Cos(\Gamma, \Lambda, X)$ has vertex set $\{g\Lambda;\ g \in \Gamma\}$, and there is an arc from $g\Lambda$ to $h\Lambda$ if $h\Lambda = gx\Lambda$, for some $x \in X$. It is an easy exercise to prove that Cayley coset graphs are well defined, $|X|$-diregular, connected, and vertex-transitive.

For a prominent example, let $\Gamma = S_{d+1}$ be the symmetric group acting on the set $[d+1] = \{1, 2, \dots, d, d+1\}$, and let $\Lambda_k$ be the subgroup of $\Gamma$ that pointwise fixes the subset $[k] = \{1, 2, \dots, k\}$, for some $k$, $2 \le k \le d$. Further, for $2 \le i \le d+1$ let $\xi_i$ be the cyclic permutation $(i \dots 21)$, and let $X = \{\xi_i;\ 2 \le i \le d+1\}$. It can be checked that the above conditions on $\Gamma$, $\Lambda$, and $X$ are satisfied; the Cayley coset digraph $Cos(S_{d+1}, \Lambda_k, X)$ is known as a *cycle prefix digraph* (see Faber, Moore and Chen [115], and also Comellas and Fiol [91]). The cycle prefix digraphs $Cos(S_{d+1}, \Lambda_k, X)$ are $(d, k)$-digraphs of order $(d+1)!/(d+1-k)!$ and they yield most of the entries of the lower triangular part in the table of largest known vertex-transitive $(d, k)$-digraphs (see end of this subsection). In particular,

$$vt_{d,k} \ge (d+1)!/(d+1-k)! \text{ if } d \ge k \ge 3.$$

Moderate improvements of the above lower bound can be obtained by removing certain adjacencies in the cycle prefix digraphs; for details we refer to [91].

We now give an example of a composition method [91]. We say that a digraph is $k$-*reachable* if for every pair of its vertices $u, v$ there exists a directed path from $u$ to $v$ of length exactly $k$. For example, the Kautz digraphs of diameter $k$ are $(k+1)$-reachable, and the cycle prefix $(d, k)$-digraphs are $k$-reachable for all $k \ge 3$. Now, let $G$ be a digraph with vertex set $V$. Let $n \ge 2$ and $t \ge 1$ be integers. Form a new digraph $G_{n,t}$ on the vertex set $\mathcal{Z}_{tn} \times V^n$, with adjacencies given by

$$(i, v_0, \dots, v_i, \dots, v_{n-1}) \to (i+j, v_0, \dots, w_i, \dots, v_{n-1}),$$

where $j \in \{1, b\}$, for some $b \in \mathcal{Z}_{tn}$, all indices on the vertices of $G$ taken mod $n$, and $w_i$ adjacent from $v_i$ in $G$. This construction was introduced by Comellas and Fiol [91] who also proved the following result: If $G$ is a vertex-transitive $d$-diregular $k$-reachable digraph then $G_{n,t}$ is also a vertex-transitive digraph, diregular of degree $2d$, of order $nt|V|^n$, and of diameter at most $kn + \ell$, where $\ell$ is the diameter of the Cayley digraph $C(\mathcal{Z}_{tn}, \{1, b\})$. The number $b$ is then chosen to minimise $\ell$. If $j$ is restricted to assume the value 1 only, then the result is a vertex-transitive $(d, k')$-digraph of order $nt|V|^n$ and of diameter at most $(k+t)n - 1$. Both constructions yield certain record examples of vertex-transitive digraphs of diameter between 7 and 11; we refer to [91] for details.

The current largest known orders of vertex-transitive digraphs for maximum out-degree $d \le 13$ and diameter $k \le 11$ are presented in Table 5; for an earlier version of the table, see Comellas and Fiol [91]. This table can be found on the website

"http://maite71.upc.es/grup_de_grafs/grafs/taula_vsd.html"

which is updated regularly by Francesc Comellas.

| d \ k | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | **6** | **10** | **20** | **27** | **72** | **144** | **171** | **336** | **504** | **737** |
| 3 | **12** | **27** | **60** | **165** | **333** | **1 152** | **1 860** | **4 446** | **10 849** | **41 472** |
| 4 | **20** | **60** | **168** | **444** | **1 260** | **7 200** | **12 090** | **38 134** | **132 012** | **648 000** |
| 5 | **30** | **120** | **360** | **1 152** | **3 582** | **28 800** | **54 505** | **259 200** | **752 914** | **5 184 000** |
| 6 | **42** | **210** | **860** | **2 520** | **7 776** | **88 200** | **170 898** | **1 411 200** | **5 184 000** | **27 783 000** |
| 7 | **56** | **336** | **1 680** | **6 720** | **20 160** | **225 792** | **521 906** | **5 644 800** | **27 783 000** | **113 799 168** |
| 8 | **72** | **504** | **3 024** | **15 120** | **60 480** | **508 032** | **1 371 582** | **18 289 152** | **113 799 168** | **457 228 800** |
| 9 | **90** | **720** | **5 040** | **30 240** | **151 200** | **1 036 800** | **2 965 270** | **50 803 200** | **384 072 192** | **1 828 915 200** |
| 10 | **110** | **990** | **7 920** | **55 400** | **332 640** | **1 960 220** | **6 652 800** | **125 452 800** | **1 119 744 000** | **6 138 320 000** |
| 11 | **132** | **1 320** | **11 800** | **95 040** | **665 280** | **3 991 680** | **19 958 400** | **282 268 800** | **2 910 897 000** | **18 065 203 200** |
| 12 | **156** | **1 716** | **17 160** | **154 440** | **1 235 520** | **8 648 640** | **51 891 840** | **588 931 200** | **6 899 904 000** | **47 703 427 200** |
| 13 | **182** | **2 184** | **24 024** | **240 240** | **2 162 160** | **17 297 280** | **121 080 960** | **1 154 305 152** | **15 159 089 098** | **115 430 515 200** |

Table 5: The order of the largest known vertex-transitive digraphs of maximum out-degree $d$ and diameter $k$.

### 3.5.2 Cayley digraphs

Let $C_{d,k}$ and $AC_{d,k}$ be the largest order of a Cayley digraph and a Cayley digraph of an abelian group, respectively, of out-degree $d$ and diameter $k$. Very little is known about $C_{d,k}$ in general. Clearly, $C_{d,1} = M_{d,1}$, and for $k \geq 3$ we know only that $C_{d,k} \leq M_{d,k}$ but we can say a little more in the case when $k = 2$. As we know from [138], for $d \geq 3$, we have $n_{d,2} = M_{d,2} - 1$ and the unique digraph of out-degree $d$ and diameter 2 is the line digraph of the complete digraph on $d + 1$ vertices.

As in the undirected case, the study of large abelian Cayley digraphs of a given out-degree (equal to the number of elements in the generating set) and given diameter can be based on a combination of group-theoretic and geometric ideas, whose genesis and background have been explained in [107]. The starting point is again the fact that any finite abelian group $\Gamma$ with an arbitrary (not necessarily symmetric) generating set $Y = \{y_1, \ldots, y_d\}$ of size $d$ is a quotient group of the free abelian $d$-generator group $\mathcal{Z}^d$ by a subgroup $N$ (of finite index) that is the kernel of the natural homomorphism $\mathcal{Z}^d \to \Gamma$ which sends the unit vector $\mathbf{e}_i \in \mathcal{Z}^d$ onto $y_i$.

Since this time we are discussing directed graphs, and therefore in our Cayley digraphs we cannot use an inverse of a generator (unless it belongs to $Y$), in our quotient group we are allowed to use only linear combinations of the vectors $\mathbf{e}_i$ with *non-negative* integer coefficients. Therefore, for any given diameter $k$, define

$$W'_{d,k} = \{(x_1, \ldots, x_d) \in \mathcal{Z}^d;\ x_i \geq 0,\ x_1 + \ldots + x_d \leq k\}.$$

Then the Cayley digraph $C(\Gamma, Y)$ has diameter at most $k$ if and only if $W'_{d,k} + N = \mathcal{Z}^d$. This allows us to conclude that $|W'_{d,k}|$ is an upper bound on $AC_{d,k}$.

The geometric connection lies again in the fact that any subgroup $N$ of $\mathcal{Z}^d$ of finite index, with the property $W'_{d,k} + N = \mathcal{Z}^d$, determines a $d$-dimensional lattice that induces 'shifts' of the set $W'_{d,k}$ so that they completely cover the elements of $\mathcal{Z}^d$. We also recall that the index $[\mathcal{Z}^d : N] = |\Gamma|$ (which gives a lower bound on $AC_{d,k}$) is equal to the absolute value of the determinant of the $d$-dimensional matrix formed by the $d$ generating vectors of $N$. This reduces the search for bounds on $AC_{d,k}$ to interesting and deep problems in combinatorial geometry (cf. [107]).

Unlike the undirected case, an exact formula for $|W'_{d,k}|$ (which, as we know, is automatically an upper bound on $AC_{d,k}$) is a matter of easy counting and it forms the right hand side of (13) below. A lower bound is much harder to obtain, and we present here the one given in Dougherty and Faber [107], based on a deep study of lattice coverings. We give both bounds as follows: There exists a constant $c$ (not depending on $d$ and $k$) such that for any fixed $d \geq 2$ and all $k$,

$$\frac{c}{d!d(\ln d)^{1+\log_2 e}} k^d + O(k^{d-1}) \leq AC_{d,k} \leq \binom{k+d}{d}. \tag{13}$$

Note that the upper bound can be considered to be the *abelian Cayley directed Moore bound* for abelian groups with $d$-element generating sets. Once again, this differs from the Moore bound $M_{d,k}$ rather dramatically; if the number of generators $d$ is fixed and $k \to \infty$ then the right hand side of (13) has the form $k^d/d! + O(k^{d-1})$.

It should not come as a surprise that the exact values of $AC_{d,k}$ are difficult to determine. With the help of lattice tilings, Dougherty and Faber [107] (and others, mainly Wong and Coopersmith [235]) showed that

$$AC_{2,k} = |W'_{2,k}| = \lfloor (k+2)^2/3 \rfloor.$$

For $d = 3$ and $k \geq 8$, similar methods (see [107] for details and more references) yield the bounds

$$0.084k^3 + O(k^2) \leq AC_{3,k} \leq 3(k+3)^3/25.$$

A table of the current best values of $AC_{3,k}$ for $k \leq 30$ appears in [107] as well.

Large Cayley digraphs can also be obtained by lifting [25, 66, 242]. As a representative example, we briefly summarise the work of Espona and Serra [110]. Let $G$ be a connected diregular digraph of out-degree $d$ and let $\mathcal{F} = \{F_1, F_2, \ldots, F_d\}$ be a factorization of $G$ into directed factors $F_i$ of in- and out-degree 1 (that is, each $F_i$ is a union of directed cycles, covering all vertices of $G$). Each factor $F_i$ then defines, in a natural way, a permutation $\phi_i$ of the vertex set of $G$, where $\phi_i(v)$ is the vertex adjacent from $v$ in the factor $F_i$. Let $\Gamma$ be the permutation group generated by the permutations $\phi_1, \ldots, \phi_d$ and let $X = \{\phi_1, \phi_2, \ldots, \phi_d\}$. Then the Cayley digraph $C(\Gamma, X)$ is a lift of the original digraph $G$. We note that this procedure can easily be translated into the language of voltage assignments on $G$.

It was pointed out in [110] that interesting large $(d, k)$-digraphs (such as butterfly digraphs) can be obtained by the above construction applied to various factorizations of the de Bruijn digraphs. For more constructions of large Cayley digraphs of given degree and diameter, see [4].

Since bounds on the diameter of a Cayley digraph in terms of a logarithm of the order of the group are essentially the same as in the undirected case (Subsection 2.4.2), we do not discuss them here.

### 3.5.3 Digraphs on surfaces

The planar version of the degree/diameter problem for digraphs was considered by Simanjuntak and Miller [219]. They showed that a planar digraph of diameter 2 and maximum out-degree $d \geq 41$ cannot have more than $2d$ vertices and that this bound is the best possible. They conjecture that the same bound holds also for $d \leq 40$. The planar version of the degree/diameter problem for $k > 2$ is totally open. Unlike in the undirected case, directed graphs embeddable on a fixed surface, other than the sphere, have not been considered from the point of view of the degree/diameter problem.

## 3.6 Related topics

In the directed case it seems that less attention has been paid to topics closely related to the ones presented in the previous sections. We therefore only consider separately the connectivity issue, while other miscellaneous contributions are summed up in Subsection 3.6.2.

### 3.6.1 Connectivity

Homobono and Peyrat [161] considered the connectivity of the digraphs proposed by Imase and Itoh. They proved that, provided the diameter is greater than 4, the connectivity of these digraphs is $d$ if $n = k(d+1)$ and $\gcd(n,d) > 1$; and $d-1$ otherwise. Imase, Soneoka and Itoh earlier proved that the connectivity is greater than or equal to $d-1$ if the graph's diameter is greater than 4. Homobono and Peyrat's paper improves upon this result.

Imase, Soneoka and Okada [165] considered the relation between the diameter $k$ and the edge (resp., vertex) connectivity of digraphs. They found that diameter minimisation results in maximising the connectivity and that all proposed small diameter digraphs have a node connectivity either $d-1$ or $d$.

A digraph is super-$\lambda$ if every edge cut of minimum size is an edge cut isolating a vertex. Soneoka [226] proved a sufficient condition for super-$\lambda$ in terms of the diameter $k$, order $n$, minimum out-degree $\delta$ and maximum out-degree $d$. He proves that a digraph is super-$\lambda$ if $n \geq \delta((d^{k-1}-1)/(d-1)+1)+d^{k-1}$. The bounds are the best possible for digraphs with diameter 2 or 3. The sufficient conditions are satisfied by many well-known digraphs, including the de Bruijn and Kautz digraphs.

Fiol [120] considers the relation between connectivity (resp., superconnectivity) and other parameters of a digraph $G$, namely, its order $n$, minimum out-degree, maximum out-degree, diameter, and a new parameter related to the number of short walks in $G$. Maximally connected and superconnected iterated line digraphs are characterised.

Fiol and Yebra [130] showed that the Moore-like bound for strongly connected bipartite digraphs $G = (V_1 \cup V_2, A)$, $d > 1$,

$$|V_1| + |V_2| \leq 2(d^{k+1}-1)/(d^2-1) \quad \text{for } k \text{ odd;}$$
$$|V_1| + |V_2| \leq 2(d^{k+1}-d)/(d^2-1) \quad \text{for } k \text{ even}$$

is attainable only when $k$ = 2,3 or 4. The interested reader can find out more about bipartite and almost bipartite Moore digraphs in studies by Fiol, Gimbert, Gómez and

Wu [126], and Fiol and Gimbert [125]; for multipartite version, see Fiol, Gimbert and Miller [127].

Balbuena, Carmona, Fábrega and Fiol [17] showed that the connectivity, as well as arc-connectivity, of a bipartite digraph is the maximum possible, provided that $n$ is large enough.

Other papers relating the order of a digraph, its maximum out-degree and diameter (and possibly other parameters) with the connectivity and/or super-connectivity of a digraph include the studies by Fábrega and Fiol [116], Fiol [121] and Xu [237]. Along with connectivity, modified concepts of diameter were considered, such as the $k$-diameter of $k$-connected graphs studied by Xu and Xu [238], or the conditional diameter in superconnected digraphs looked at by Balbuena, Fàbrega, Marcote and Pelayo [18].

### 3.6.2 Other related problems

Fiol, Lladó and Villar [128] considered the order/degree problem for digraphs; they constructed a family of digraphs with the smallest possible diameter, given order and maximum out-degree.

Aider [1] studied bipartite digraphs with maximum in- and out-degree $d$ ($> 1$) and diameter $k$. He showed that the order of such a digraph is at most

$$2\frac{d^{k+1}-1}{d^2-1} \qquad \text{if } k \text{ is odd;}$$

$$2\frac{d^{k+1}-d}{d^2-1} \qquad \text{if } k \text{ is even.}$$

The author then finds some pairs $d$ and $k$, for which there exist bipartite digraphs of the given order ('bipartite Moore digraphs') and some pairs for which there are no such bipartite digraphs. Additionally, a variety of properties of such digraphs are established.

Gómez, Morillo and Padró [150] consider $(d, k, k', s)$-digraphs (digraphs with maximum out-degree $d$ and diameter $k$ such that, after the deletion of any $s$ of its vertices, the resulting digraph has diameter at most $k'$). The authors' goal is to find such bipartite digraphs with order as large as possible. They give new families of digraphs satisfying a Menger-type condition, namely, between any pair of non-adjacent vertices there are $s+1$ internally disjoint paths of length at most $k'$, and they obtain new families of bipartite $(d, k, k', s)$-digraphs with order very close to the upper bound.

Munoz and Gómez [192] continued this research and obtained new families of asymptotically optimal $(d, k, k', s)$-digraphs.

Morillo, Fiol and Fábrega [190], Morillo, Fiol and Yebra [191], Comellas, Morillo and

Fiol [94] used plane tessellations to construct families of bipartite digraphs of degree two and with maximum order, minimum diameter, and minimum mean distance, defined by $\bar{k} = \sum_{i,j \in V} d_{ij}/n^2$. The last parameter was also studied earlier by Wong [234], who considered a subclass of digraphs in which the number of nodes is $n$ and diameter is $k$; he showed that the minimum values of diameter and the average distance are both of the order of $d^d\sqrt{n}$.

Knor [172] defines *radially Moore digraphs* as regular digraphs with radius $s$, diameter $k \leq s+1$ and the maximum possible number of nodes. He shows that, for every $s$ and $d$, there exists a regular radially Moore digraph of degree $d$ and radius $s$. He also gives an upper bound on the number of central nodes in a radially Moore digraph of degree 2.

Unilaterally connected digraphs were studied from the perspective of the degree/diameter problem by Gómez, Canale and Munoz [143, 144].

*Partially directed Moore graphs* (also called *mixed Moore graphs*) were introduced and investigated by Bosák [62]. Bosák [62] defines a partially directed Moore graph as a simple, finite and homogeneous (each vertex is an endpoint of $r$ undirected (two-way) edges and is an origin and a terminal of $z$ directed (one-way) edges, where $r$ and $z$ are independent of the choice of a vertex) graph satisfying the condition: There exists exactly one trail (an edge can be used only once and wrong way is not allowed) from any vertex $u$ to any vertex $v$ of length at most the diameter. Bosák found divisibility conditions concerning the distribution of undirected and directed edges in mixed Moore graphs of diameter 2, and he produced some examples of mixed Moore graphs.

This line of research was continued by Nguyen, Miller and Gimbert [198] who proved the equivalence of mixed tied graphs and mixed Moore graphs. See also Nguyen and Miller [196]. It is shown that all proper mixed tied graphs are totally regular and that they do not exist in the case when the diameter is greater than 2. They also proved that all the known mixed Moore graphs of diameter 2 are unique.

# 4 Conclusion

In this survey we have presented results and research directions concerning the degree/diameter problem.

Here we give a list of some of the open problems in this area.

1. Does there exist a Moore graph of diameter 2 and degree 57?
2. At present we have only non-diregular examples of a digraph with $n = 49$, $d = 2$ and $k = 5$. Does there exist a diregular version of a digraph with the same parameters $n, d, k$?
3. Is $n_{d,k}$ monotonic in $d$ and/or in $k$?
4. Find graphs (resp., digraphs) which have larger number of vertices than the currently largest known graphs (resp., digraphs).
5. In particular, does the largest graph of diameter 2 and degree 6 have 32 or 33 vertices?
6. Answer the question of Bermond and Bollobás (end of Section 2.2), which asks if, for each integer $c > 0$, there exist $\Delta$ and $D$, such that $n_{\Delta,D} \leq M_{\Delta,D} - c$.
7. Prove or disprove the conjecture of Bollobás (Subsection 2.3.1), stating that for each $\varepsilon > 0$, $n_{\Delta,D} > (1-\varepsilon)\Delta^D$, for sufficiently large $\Delta$ and $D$.
8. Is it true that $n_{\Delta,D} = vt_{\Delta,D}$, for infinitely many pairs of $\Delta \geq 3$ and $D \geq 2$?
9. Does there exist a radially Moore undirected graph for every diameter and degree?
10. Is there a mixed Moore graph of order 40, diameter 2, and such that each vertex is incident with 3 undirected edges and each vertex is the starting point of 3 directed arcs?
11. Prove the diregularity or otherwise of digraphs close to Moore bound for defect greater than one.
12. Prove or disprove the following generalisation of the result of Knor and Širáň from Subsection 2.4.5: For each surface $\mathcal{S}$ and for each $D \geq 2$, there exist a constant $\Delta_{\mathcal{S}}$ such that for each $\Delta \geq \Delta_{\mathcal{S}}$, we have $n_{\Delta,D}(\mathcal{S}) = n_{\Delta,D}(\mathcal{S}_0)$.
13. Motivated by the result of Šiagiová and Simanjuntak (Subsection 2.4.5), investigate the existence of the limit of $n_{\Delta,D}(\mathcal{S})/\Delta^{\lfloor D/2 \rfloor}$ as $\Delta \to \infty$.

In conclusion, we would like to comment briefly on the relationships between the three parameters that have featured heavily in this survey; namely, order, degree and diameter. Throughout this survey, we have considered the degree/diameter problem, that is, maximising the order of a graph, resp., digraph. However, considering the three parameters of a graph: order, degree and diameter, there are two additional extremal problems that arise if we optimise in turn each one of the parameters while holding the other two parameters fixed, namely,

- *Order/degree problem*: Given natural numbers $n$ and $\Delta$, find the smallest possible diameter $D_{n,\Delta}$ in a graph of order $n$ and maximum degree $\Delta$.
- *Order/diameter problem*: Given natural numbers $n$ and $D$, find the smallest possible maximum degree $\Delta_{n,D}$ in a graph of order $n$ and diameter $D$.

The statements of the directed version of the problems differ only in that 'degree' is replaced by 'out-degree.

For both undirected and directed cases, most of the attention has been given to the degree/diameter problem, some attention has been received by the order/degree problem but the order/diameter problem has been largely overlooked so far. For more details concerning the three problems and their relationships, see [181, 188, 183].

Although we tried to include all references to the degree/diameter problem and other research related to the Moore bounds, it is quite likely that we have accidentally or out of ignorance left out some references that should have been included. We apologise for any such oversights. Fortunately, this is a dynamic survey and we will be updating it periodically. We will very much appreciate finding out about any omissions, as well as new results in the degree/diameter problem and related topics.

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# References

[1] M. Aider, Réseaux d'interconnexion bipartis orientés, *Rev. Maghrébine Math.* **1** (1992) No. 1, 79–92.

[2] N. Alon, S. Hoory and N. Linial, The Moore bound for irregular graphs, *Graphs and Combinatorics* **18** (2002) 53–57.

[3] F. Aguiló-Gost, New dense families of triple loop networks. *Discrete Math.* **197-198** (1999) 15–27.

[4] F. Aguiló, M.A. Fiol and C. Garcia, Triple loop networks with small transmission delay, *Discrete Mathematics* **167/168** (1997) 3–16.

[5] S.B. Akers, On the construction of $(d, k)$ graphs, *IEEE Trans. Electronic Computers* (Short Notes) **EC-14** (1965) 448.

[6] S.B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, *IEEE Transactions on Computers* **38** (1989) 555–566.

[7] I. Alegre, personal communication to M. Miller via K. Culik II and I. Fris.

[8] I. Alegre, M.A. Fiol and J.L.A. Yebra, Some large graphs with given degree and diameter, *J. Graph Theory* **10** (1986) 219–224.

[9] J. Allwright, New $(\Delta, D)$-graphs discovered by heuristic search, *Discrete Applied Mathematics* **37/38** (1992) 3–8.

[10] F. Annexstein, M. Baumslag, A. L. Rosenberg, Group action graphs and parallel architectures, *SIAM J. Comput.* **19** (1990) 544–569.

[11] B. W. Arden and H. Lee, Analysis of chordal ring network, *IEEE Trans. Comput.* **30** (1981) No. 4, 291–295.

[12] L. Babai, P. Erdős, Representation of group elements as short products, *"Theory and practice of combinatorics"*, North Holland Math. Stud. 60, North Holland, Amsterdam, 1982, 27–30.

[13] L. Babai, G. L. Hetyei, On the diameter of random Cayley graphs of the symmetric group, *Combin. Probab. Comput.* **1** (1992) No. 3, 201–208.

[14] L. Babai, W. M. Kantor, A. Lubotzky, Small-diameter Cayley graphs for finite simple groups, *European J. Combin.* **10** (1989) No. 6, 507–522.

[15] L. Babai, A. Seress, On the diameter of Cayley graphs of the symmetric group, *J, Combinat. Theory Ser A* **49** (1988) No. 1, 175–179.

[16] L. Babai, A. Seress, On the diameter of permutation groups, *European J. Combinat.* **13** (1992) No. 4, 231–243.

[17] M.C. Balbuena, A. Carmona, J. Fábrega and M.A. Fiol, On the order and size of $s$-geodetic digraphs with given connectivity, *Discrete Mathematics* **174** (1997) 19–27.

[18] M.C. Balbuena, J. Fàbrega, X. Marcote, I. Pelayo, Superconnected digraphs and graphs with small conditional diameters, *Networks* **39** (2002) No. 3, 153-160.

[19] E. Bannai and T. Ito, On finite Moore graphs, *J. Fac. Sci. Tokyo Univ.* **20** (1973) 191–208.

[20] E. Bannai and T. Ito, Regular graphs with excess one, *Discrete Mathematics* **37** (1981) 147–158.

[21] D. Barth and M.-C. Heydemann, A new digraph composition with applications to de Bruijn and generalised de Bruijn digraphs, *Discrete Applied Mathematics* **77** (1997) 99–118.

[22] R. Bar-Yehuda and T. Etzion, Connections between two cycles – a new design of dense processor interconnection networks, *Discrete Applied Mathematics* **37/38** (1992) 29–43.

[23] E.T. Baskoro, *Optimal Directed Networks*, Master Thesis, UNE, Armidale, Australia (1991).

[24] E.T. Baskoro and Amrullah, Almost Moore digraphs of degree 4 without selfrepeats, *Journal of the Indonesian Mathematical Society (MIHMI)*, **8** (2002) No. 1, 51 - 60.

[25] E.T. Baskoro, L. Branković, M. Miller, J. Plesník, J. Ryan and J. Širáň, Large digraphs with small diameter: A voltage assignment approach, *Journal of Combinatorial Mathematics and Combinatorial Computing* **24** (1997) 161–176.

[26] E.T. Baskoro, Y.M. Cholily and M. Miller,Structure of repeat cycles in almost Moore digraph with selfrepeats and diameter 3, preprint.

[27] E.T. Baskoro, Y.M. Cholily and M. Miller, Structure of repeat cycles in almost Moore digraphs, preprint.

[28] E.T. Baskoro and M. Miller, On the construction of networks with minimum diameter, *Australian Computer Science Communications* **C 15** (1993) 739–743.

[29] E.T. Baskoro and M. Miller, A procedure for constructing a minimum-diameter digraph, *Bulletin of ICA* **17** (1996) 8–14.

[30] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Regular digraphs of diameter 2 and maximum order, *the Australasian Journal of Combinatorics* **9** (1994) 291–306. See also Errata, **13** (1995).

[31] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to Moore bound, *J. Graph Theory* **20** (1995) 339–349.

[32] E.T. Baskoro, M. Miller and J. Plesník, On the structure of digraphs with order close to the Moore bound, *Graphs and Combinatorics* **14** (1998) No.2, 109–119.

[33] E.T. Baskoro, M. Miller and J. Plesník, Further results on almost Moore digraphs, *Ars Combinatoria* **56** (2000) 43–63.

[34] E.T. Baskoro, M. Miller, J. Širáň and M. Sutton, A complete characterisation of almost Moore digraphs of degree three, *J. Graph Theory*, to appear.

[35] E.T. Baskoro, M. Miller and J. Širáň, A remark on almost Moore digraphs of degree three, *Acta Math. Univ. Comenianae* **LXVI** (1997) No.2, 285–291.

[36] D. Bauer, F.T. Boesch, C. Suffel and R.Van Slyke, On the validity of a reduction of reliable networks design to a graph extremal problem, *IEEE Trans. on Circuits and Systems* **CAS-34** (1987) 1579–1581.

[37] C.T. Benson, Minimal regular graphs of girth eight and twelve, *Canad. J. Math* **18** (1966) 1091–1094.

[38] C.T. Benson and N.E. Losey, On a graph of Hoffman and Singleton, *J. Combinatorial Theory Ser. B* **11** (1971) 67–79.

[39] J.-C. Bermond and B. Bollobás, The diameter of graphs: A survey, *Congressus Numerantium* **32** (1981) 3–27.

[40] J.-C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer networks: a survey *J. Parallel and Distributed Computing* **24** (1995) 2–10.

[41] J.-C. Bermond, C. Delorme and G. Farhi, Large graphs with given degree and diameter III, *Annuals of Discrete Mathematics* **13** (1982) 23–32.

[42] J.-C. Bermond, C. Delorme and G. Farhi, Large graphs with given degree and diameter II, *Journal of Combinatorial Theory* (B) **36** (1984) 32–48.

[43] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Grands graphes non dirigés de degré et diamètre fixés, *Ann. Discrete Math.* **17** (1982) 65–73.

[44] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Tables of large graphs with given degree and diameter, *Information Processing Letters* **15** (1982) 10–13.

[45] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Grands graphes de degré et diametre fixés, *Combinatorial Mathematics* (Marseille-Luminy, 1981) 65–73; *North-Holland Math. Stud.* **75** North-Holland, Amsterdam-New York (1983).

[46] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Tables of large $(\Delta, D)$-graphs, *Discrete Applied Mathematics* **37/38** (1992) 575–577.

[47] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Strategies for interconnection networks: some methods from graph theory, *Journal of Parallel and Distributed Computing* **3** (1986) 433–449.

[48] J.-C. Bermond, F.O. Ergincan, Bus interconnection networks, Discrete Appl. Math. **68** (1996) 1–15.

[49] J.-C. Bermond, P. Hell, On even factorizations and the chromatic index of the Kautz and de Bruijn digraphs, *J. Graph Theory* **17** (1993) 647–655.

[50] J.-C. Bermond, P. Hell, J.-J. Quisquater, Construction of large packet radio networks, *Parallel Processing Letters* **2** (1992) 3–12.

[51] J.-C. Bermond, N. Homobono and C. Peyrat, Large fault-tolerant interconnection networks, *Graphs and Combinatorics* **5** (1989) 107–123.

[52] J.-C. Bermond, N. Homobono and C. Peyrat, Connectivity of Kautz networks, *Discrete Mathematics* **114** (1993) No. 1-3, 51–62.

[53] J.-C. Bermond, C. Peyrat, De Bruijn and Kautz networks, a competitior for the hypercube ? *Proc. 1st European Workshop on Hypercubes and Distributed Computers*, Rennes (1989) 279–293.

[54] N.I. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Second Edition, Great Britain (1993).

[55] N.I. Biggs, Girth, valency and excess, *Linear Algebra Appl.* **31** (1980) 55–59.

[56] N.I. Biggs and T. Ito, Graphs with even girth and small excess, *Math. Proc. Cam-

*bridge Phil. Soc.* **88** (1980) No. 1, 1–10.

[57] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York (1978).

[58] J. Bond and C. Delorme, New large bipartite graphs with given degree and diameter, *Ars Combinatoria* **25C** (1988) 123–132.

[59] J. Bond and C. Delorme, A note on partial Cayley graphs, *Discrete Mathematics* **114** (1993) No.1-3, 63–74.

[60] J. Bosák, Cubic Moore graphs, *Mat. Časopis Slovenskej Akadémie Vied* **20** (1970) 72–80.

[61] J. Bosák, Directed graphs and matrix equations, *Math. Slovaca* **28** (1978) No. 2, 189–202.

[62] J. Bosák, Partially directed Moore graphs, *Math. Slovaca* **29** (1979) No. 2, 181–196.

[63] J. Bosák, A. Kotzig and Š. Znám, Strongly geodetic graphs, *Journal of Combinatorial Theory* **5** (1968) 170–176.

[64] R.C. Bose and T.A. Dowling, A generalization of Moore graphs of diameter two, *J. Comb. Th.* **11** (1971) 213–226.

[65] L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, Large graphs with small degree and diameter: A voltage assignment approach, *Australas. J. Combin.* **18** (1998) 65–76.

[66] L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, A note on constructing large Cayley graphs of given degree and diameter by voltage assignments, *Electronic J. Combin.* **5** (1998) No. 1, article R9.

[67] W.G. Bridges and S. Toueg, On the impossibility of directed Moore graphs, *J. Combinatorial Theory B* **29** (1980) 339–341.

[68] W.G. Brown, On graphs that do not contain a Thompsen graph, *Canad. Math. Bull.* **9** (1966) 281–285.

[69] N.G. de Bruijn, A combinatorial problem, *Koninkl. Nederl. Acad. Wetensch. Proc. Ser* A **49** (1946) 758–764.

[70] R.W. Buskens and R.G. Stanton, Generalized Moore graphs on twelve and thirteen vertices, *Ars Combinat.* **23B** (1987) 87–132.

[71] R.W. Buskens, M.J. Rogers and R.G. Stanton, A census of tetravalent generalized Moore networks, *Congressus Numerantium* **52** (1986) 255–296.

[72] P.J. Cameron, Automorphism groups of graphs, in: *Selected Topics in Graph Theory* **2** (1983) Acad. Press.

[73] L. Campbell, Dense group networks, *Discrete Applied Mathematics* **37/38** (1992) 65–71.

[74] L. Campbell, G.E. Carlsson, M.J. Dinneen, V. Faber,

[75] E.A. Canale and J. Gómez, Asymptotically large $(\Delta, D)$-graphs, *Discrete Applied Math.*, to appear.

[76] E.A. Canale and J. Gómez, Superfluous edges and exponential expansions of deBruijn and Kautz digraphs, *Discrete Applied Math.* **138** (2004), 303-331.

[77] M.R. Fellows, M.A. Langston, J.W. Moore, A.P. Mullhaupt and H.B. Sexton, Small diameter symmetric networks from linear groups, *IEEE Transactions on Computers* **41** (1992) No. 2, 218–220.

[78] G.E. Carlsson, J. Cruthirds, H. Sexton and C. Wright, Interconnection networks based on generalization of cube-connected cycles, *IEEE Trans. Comput.* **C-34** (1985) No. 8, 769–772.

[79] V.G. Cerf, D.D. Cowan, R.C. Mullin and R.G. Stanton, Computer networks and generalized Moore graphs, *Congr. Num.* **9** (1973) 379–398.

[80] V.G. Cerf, D.D. Cowan, R.C. Mullin and R.G. Stanton, Trivalent generalized Moore networks on sixteen nodes, *Utilitas Math.* **6** (1974) 259–283.

[81] Y.M. Cholily, E.T. Baskoro and S. Uttunggadewa, Some conditions for the existence of $(d, k)$-digraphs containing selfrepeat, preprint.

[82] D.V. Chudnovsky, G.V. Chudnovsky and M.M. Denneau, Regular graphs with small diameter as models for interconnection networks, *Proc. 3rd Internat. Conf. on Supercomputing ICS'88*, Boston (1988) 232–239.

[83] F.R.K. Chung, Diameters of communication networks, in: *AMS Short Course on the Mathematics of Information Processing* **34**, Amer. Math. Soc., Providence, RI (1986) 1–18.

[84] F.R.K. Chung, Diameters of graphs: old problems and new results, *Congr. Numer.* **60** (1987) 295–317.

[85] F.R.K. Chung, Diameters and eigenvalues, *Journal of American Mathematical Society* **2** (1989) No. 2, 187–196.

[86] F.R.K. Chung, C. Delorme and P. Solé, Multidiameters and multiplicities, *European J. Combin.* **20** (1999) 629–640.

[87] Chung, F.R.K.; Faber, V.; Manteuffel, Thomas A. An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian. *SIAM J. Discrete Math.* **7** (1994) No.3, 443–457.

[88] F. Comellas, The degree/diameter problem for graphs, *http://www-mat.upc.es/grup.de.grafs/table.g.html.*

[89] F. Comellas, *http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.htm.*

[90] F. Comellas and M.A. Fiol, Using simulated annealing to design interconnection networks, *Technical Report* DMAT-05-0290, Universitat Politécnica de Catalunya (1990).

[91] F. Comellas and M.A. Fiol, Vertex symmetric digraphs with small diameter, *Discrete Applied Mathematics* **58** (1995) 1–12.

[92] F. Comellas, M.A. Fiol and J. Gómez, On large vertex symmetric 2-reachable digraphs, *Parallel Processing Letters* **4** (1994) 379–384.

[93] F. Comellas and J. Gómez, New large graphs with given degree and diameter, *Graph Theory, Combinatorics and Algorithms* **1,2**: Proc. 7th Quadrennial Int'l Conf. on the theory and Appl. of Graphs, Kalamazoo (MI, USA) (1992), edited by Y. Alavi and A. Schwenk, (1995) 221–233.

[94] F. Comellas, P. Morillo and M.A. Fiol, Diameter and mean distance of bipartite digraphs related to plane tessellations, *Ars Combin.* **29** B (1990) 245–254.

[95] R.M. Damerell, On Moore graphs, *Proc. Cambridge Phil. Soc.* **74** (1973) 227–236.

[96] R.M. Damerell, On Moore geometries II, *Math. Proc. Camb. Phil. Soc.* **90** (1981) 33–40.

[97] R.M. Damerell and M.A. Georgiacodis, On Moore geometries I, *J. London Math. Soc.* **23** (1981) No. 2, 1–9.

[98] C. Delorme, Grands graphes de degré et diamètre donnés, *European J. Combin.* **6** (1985) 291–302.

[99] C. Delorme, Large bipartite graphs with given degree and diameter, *J. Graph Theory* **8** (1985) 325–334.

[100] C. Delorme, Examples of products giving large graphs with given degree and diameter, *Discrete Applied Mathematics* **37/38** (1992) 157–167.

[101] C. Delorme and G. Farhi, Large graphs with given degree and diameter-part I, *IEEE Transactions on Computers* **C-33** (1984) 857–860.

[102] C. Delorme and J. Gómez, Some new large compound graphs, *Europ. J. of Combin.* **23** (2002) No. 5, 539–547.

[103] C. Delorme and J.J. Quisquater, Some new large compound graphs, *Internal Report* **317**, LRI, December 1986.

[104] M.J. Dinneen, *Algebraic methods for efficient network constructions*, Master's Thesis, Department of Computer Science, University of Victoria, Canada (1991).

[105] M.J. Dinneen and P.R. Hafner, New results for the degree/diameter problem, *Networks* **24** (1994) 359–367.

[106] K.W. Doty, New designs for dense processor interconnection networks, *IEEE on Transactions on Computers* **C-33** (1984) No. 5, 447–450.

[107] R. Dougherty and V. Faber, The degree-diameter problem for several varieties of Cayley graphs, I: The Abelian case, *preprint* (2001).

[108] R.D. Dutton and R.C. Brigham, Edges in graphs with large girth, *Graphs and Combinatorics* **7** (1991) 315–321.

[109] B. Elspas, Topological constraints on interconnection-limited logic, *Proc. IEEE Fifth Symposium on Switching Circuit Theory and Logical Design*, IEEE **S–164** (1964) 133-147.

[110] M. Espona and O. Serra, Cayley digraphs based on the networks, *SIAM J. Discrete Math.* **11** (1998) No. 2, 305–317.

[111] P. Erdős, S. Fajtlowicz and A.J. Hoffman, Maximum degree in graphs of diameter

2, *Networks* **10** (1980) 87–90.

[112] G. Exoo, A family of graphs and the degree/diameter problem, *J. Graph Theory* **37** (2001) No. 2, 118–124.

[113] G. Exoo, Voltage graphs, group presentations and cages, *Electron. J. Combinat.* **11** (2004) Note 2, 7pp.

[114] V. Faber and J.W. Moore, High-degree low-diameter interconnection networks with vertex symmetry: the directed case. *Technical Report* LA-UR-88-1051, Los Alamos National Laboratory, Los Alamos, New Mexico (1988).

[115] V. Faber, J.W. Moore and W.Y.C Chen, Cycle prefix digraphs for symmetric interconnection networks, *Networks* **23** (1993) 641–649.

[116] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, *Journal of Graph Theory* **13** (1989) 657-668.

[117] S. Fajtlowicz, Graphs of diameter 2 with cyclic defect, *Colloquium Mathematicum* **51** (1987) 103–106.

[118] M. Fellows, P. Hell and K. Seyffarth, Large planar graphs with given diameter and maximum degree, *Discrete Appl. Mathematics* **61** (1995) 133–153.

[119] M. Fellows, P. Hell and K. Seyffarth, Constructions of large planar networks with given degree and diameter, *Networks* **32** (1998) 275–281.

[120] M.A. Fiol, Connectivity and superconnectivity of large graphs and digraphs, Twelfth British Combinatorial Conference (Norwich, 1989), *Ars Combinatoria* **29B** (1990) 5–16.

[121] M.A. Fiol, The connectivity of large digraphs and graphs, *J. Graph Theory* **17** (1993) 31–45.

[122] M.A. Fiol, I. Alegre and J.L.A. Yebra, Line digraph iterations and the $(d, k)$ problem for directed Graphs, *IEEE Transactions on Computers* **C-33** (1984) 400–403.

[123] M.A. Fiol, J. Fàbrega and M. Escudero, Short paths and connectivity in graphs and digraphs, *Ars Combinatoria* **29B** (1990) 17–31.

[124] M.A. Fiol and J. Fàbrega, Some compound graphs, *Stochastica* **7** No.2 (1983) 137–143.

[125] M.A. Fiol and J. Gimbert, On almost Moore bipartite digraphs with odd diameter, *Congr. Numer.* **152** (2001), 169-187.

[126] M.A. Fiol, J. Gimbert, J. Gómez and Y. Wu, On Moore bipartite digraphs, *J. Graph Theory* **43** (2003) No. 3, 171-181.

[127] M.A. Fiol, J. Gimbert and M. Miller, Multipartite Moore digraphs, preprint.

[128] M.A. Fiol, A.S. Lladó and J.L. Villar, Digraphs on alphabets and the $(d, N)$ digraph problem, *Ars Combinatoria* **25C** (1988) 105–122.

[129] M.A. Fiol and A.S. Lladó, The partial line digraph technique in the design of large interconnection networks, *IEEE Transactions on Computers* **41** (1992) 848–857.

[130] M.A. Fiol and J.L.A. Yebra, Dense bipartite digraphs, *J. Graph Theory* **14** (1990) 687–700.

[131] M. A. Fiol, J. L. A. Yebra, J. Fabrega, Sequence graphs and interconnection networks, *Ars Combinatoria* **16A** (1983) 7–13.

[132] M. Fried and J.H. Smith, Primitive groups, Moore graphs, and rational curves, *Michigan Math. J.* **19** (1972) 341–346.

[133] H. Friedman, A design for $(d, k)$ Graphs, *IEEE Trans. Electronic Computers* **EC-15** (1996) 253–254.

[134] H. Friedman, On the impossibility of certain Moore graphs, *J. Combinatorial Theory Ser. B* **10** (1971) 245–252.

[135] F.J. Fuglister, On finite Moore geometries, *J. Comb. Th.* A **23** (1977) 187–197.

[136] F.J. Fuglister, The nonexistence of Moore geometries of diameter 4, *Discrete Mathematics* **45** (1983) 229238.

[137] C. Garcia and C. Peyrat, Large Cayley graphs on an abelian group, *Discrete Appl. Mathematics* **75** (1997) No. 2, 125–133.

[138] J. Gimbert, On the existence of $(d, k)$-digraphs, 16th British Combinatorial Conference (London, 1997), *Discrete Mathematics* **197/198** (1999) 375–391.

[139] J. Gimbert, On digraphs with unique walks of closed lengths between vertices, *Australasian Journal of Combinatorics* **20** (1999) 77–90.

[140] J. Gimbert, Enumeration of almost Moore digraphs of diameter two, *Discrete Math.* **231** (2001) 177–190.

[141] J. Gómez, Generalized compound graphs, *Ars Combinatoria* **29B** (1990) 33–53.

[142] J. Gómez, On large $(\Delta, 6)$-graphs *Networks*, to appear.

[143] J. Gómez, E.A. Canale and X. Munoz, On the unilateral $(\Delta, D^*)$-problem, *Networks* **34** (2000), 164-171.

[144] J. Gómez, E.A. Canale and X. Munoz, Unilaterally connected large digraphs and generalized cycles, *Networks* **42** (2000), 181-188.

[145] J. Gómez and M. Escudero, Optimal edge coloring of large graphs, *Networks* **34** (1999) 61–65.

[146] J. Gómez and M.A. Fiol, Dense compound graphs, *Ars Combinatoria* **20A** (1985) 211–237.

[147] J. Gómez, M.A. Fiol and O. Serra, On large $(\Delta, D)$-graphs, *Discrete Mathematics* **114** (1993) 219–235.

[148] J. Gómez, M.A. Fiol and J.L.A. Yebra, Graphs on alphabets as models for large interconnection networks, *Discrete Applied Mathematics* **37/38** (1992) 227–243.

[149] J. Gómez and M. Miller, Two new families of compound graphs, preprint.

[150] J. Gómez, P. Morillo and C. Padró, Large $(d, D, D', s)$-bipartite digraphs, *Discrete Applied Mathematics* **59** (1995) No. 2, 103–114.

[151] J. Gómez, C. Padró and S. Perennes, Large generalized cycles, *Discrete Applied Mathematics* **89** (1998) 107–123.

[152] J. Gómez, I. Pelayo and C. Balbuena, New large graphs with given degree and diameter six, *Networks* **34** (1999) No. 2, 154–161.

[153] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York (1987).

[154] W. H. Haemers, *Eigenvalue technique in design and graph theory*, Mathematical Centre Tracts **121**, Matematisch Zentrum, Amsterdam (1980).

[155] P. R. Hafner, Large Cayley graphs and digraphs with small degree and diameter, *Computational Algebra and Number Theory* (W. Bosma and van der Poorten, eds.) Kluwer, Amsterdam (1995) 291–302.

[156] P. R. Hafner, On the graphs of McKay-Miller-Širáň, *Proc. 8th Intl. Colloquium on Structural Information and Communication Complexity (SIROCCO 8), Proceedings in Informatics* **11** (F. Comellas, J. Fabrega, P. Fraigniaud, eds.), Carleton Scientific (2001) 209–215.

[157] P. R. Hafner, The Hoffman-Singleton graph and its automorphisms, *J. Algebraic Combin.*, to appear.

[158] P. R. Hafner, Geometric realisation of the automorphism group of the graphs of McKay-Miller-Širáň, preprint.

[159] P. Hell and K. Seyffarth, Largest planar graphs of diameter two and fixed maximum degree, *Discrete Mathematics* **111** (1993) 313–332.

[160] A.J. Hoffman and R.R. Singleton, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* **4** (1960) 497–504.

[161] N. Homobono and C. Peyrat, Connectivity of Imase and Itoh digraphs, *IEEE TRansactions on Computers* **37** (1988) 1459–1461.

[162] M. N. Huxley, On the difference between consecutive primes, *Invent. Math.* **15** (1972) 164–170.

[163] M. Imase and M. Itoh, Design to minimize diameter on building-block network, *IEEE Trans. on Computers* **C-30** (1981) 439–442.

[164] M. Imase and M. Itoh, A design for directed graphs with minimum diameter, *IEEE Trans. on Computers* **C-32** (1983) 782–784.

[165] M. Imase, T. Soneoka and K. Okada, Connectivity of regular directed graphs with small diameter, *IEEE Transactions on Computers* **C-34** (1985) 267–273.

[166] H. Iswadi and E.T. Baskoro, On (4,2)-digraphs containing a cycle of length 2, *Bull. Malaysian Math. Sci. Soc.* (2) **23** (2000), No. 1, 79–91.

[167] L.O. James, A combinatorial proof that the Moore (7,2) graph is unique, *Utilitas Math.* **5** (1974) 79–84.

[168] L.K. Jorgensen, Diameters of cubic graphs, *Discrete Appl. Mathematics* **37/38** (1992) 347–351.

[169] W. M. Kantor, Some large trivalent graphs having small diameters, *Discrete Applied*

*Math.* **37/38** (1992), 353–357.

[170] W.H. Kautz, Bounds on directed $(d, k)$ graphs, *Theory of cellular logic networks and machines*, AFCRL-68-0668 Final Report (1968) 20–28.

[171] W.H. Kautz, Design of optimal interconnection networks for multiprocessors, *Architecture and Design of Digital Computers*, NATO Advanced Summer Institute (1969) 249–272.

[172] M. Knor, A note on radially Moore digraphs, *IEEE Trans. Comput.* **45** (1996) 381–383.

[173] M. Knor and J. Širáň, Extremal graphs of diameter two and given maximum degree, embeddable in a fixed surface, *Journal of Graph Theory* **24** (1997) 1–8.

[174] I. Korn, On $(d, k)$ graphs, *IEEE Trans. Electronic Computers* **EC-16** (1967) 90.

[175] F. Kramer, Relations between the diameter of a graph and the degrees of the nodes of a graph (Romanian), *Rev. Anal. Numer. Teoria Aproximaţiei* **1** (1972) 125–131.

[176] K. Kurosawa and S. Tsujii, Considerations on diameter of communication networks, *Electronics and Communications in Japan* **64A** (1981) No. 4, 37–45.

[177] S. Lakshmivarahan, J.S. Jwo and S.K. Dhall, Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey, *Parallel Computing* **19** (1993) 361–407.

[178] R. Lipton and R. E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979) No. 2, 177–189.

[179] B.D. McKay, M. Miller and J. Širáň, A note on large graphs of diameter two and given maximum degree, *J. Combin. Theory Ser. B* **74** (1998) No. 1, 110–118.

[180] G. Memmi, and Y. Raillard, Some new results about the $(d, k)$ graph problem, *IEEE Trans. on Computers* **C-31** (1982) 8.

[181] M. Miller, *Extremal Digraphs*, M.A. Thesis, Department of Mathematics, Statistics and Computing Science, University of New England, Australia (1986).

[182] M. Miller, Digraph covering and its application to two optimization problems for digraphs, *Australasian Journal of Combinatorics* **3** (1991) 151–164.

[183] M. Miller, Some open problems in optimal digraphs, *Tatra Mountains Mathematical Publications* **9** (1996) 89–96.

[184] M. Miller and I. Fris, Maximum order digraphs for diameter 2 or degree 2, *Pullman volume of Graphs and Matrices*, *Lecture Notes in Pure and Applied Mathematics* **139** (1992) 269–278.

[185] M. Miller, J. Gimbert, J. Širáň and Slamin, Almost Moore digraphs are diregular, *Discrete Mathematics* **218** (2000) 265–270.

[186] M. Miller, M.H. Nguyen and R. Simanjuntak, Repeat structure in regular graphs and digraphs, *Proceedings of the 2nd European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB2003), Prague* (2003) 265–270.

[187] M. Miller and J. Širáň, Digraphs of degree two which miss the Moore bound by two,

*Discrete Mathematics* **226** (2001) 269–274.

[188] M. Miller and Slamin, On the monotonicity of minimum diameter with respect to order and maximum out-degree, *Proceedings of COCOON 2000, Lecture Notes in Computer Science* **1858** (D.-Z. Du, P. Eades, V. Estivill-Castro, X. Lin (eds.)) (2000) 192–201.

[189] M. Miller and R. Simanjuntak, Graphs of order two less than the Moore bound, *Discrete Mathematics*, to appear.

[190] P. Morillo, M.A. Fiol and J. Fábrega, The diameter of directed graphs associated to plane tessellations, *Ars Combin.* **20** A (1985) 17–27.

[191] P. Morillo, M.A. Fiol and J.L.A. Yebra, A new family of graphs associated with plane tessellations, *Ars Combin.* **20** A (1985) 163–169.

[192] X. Munoz and J. Gómez, Asymptotically optimal $(\Delta, D', s)$-digraphs, *Ars Combin.* **49** (1998) 97–111.

[193] M.H. Nguyen and M. Miller, Structural properties of graphs of diameter 2 with maximal repeats, preprint.

[194] M.H. Nguyen and M. Miller, Towards a characterization of graphs of diameter 2 and defect 2, preprint.

[195] M.H. Nguyen and M. Miller, The nonexistence of certain graphs of diameter 2 and defect 2, preprint.

[196] M.H. Nguyen and M. Miller, Moore bound for mixed networks, preprint.

[197] M.H. Nguyen and M. Miller, A classification of graphs of diameter 2 and defect 3, preprint.

[198] M.H. Nguyen, M. Miller and J. Gimbert, On mixed Moore graphs, *Discrete Mathematics*, to appear.

[199] J. Opatrný, D. Sotteau, Linear congruential graphs, *Graph theory, combinatorics, algorithms, and applications* (San Francisco, CA, 1989), SIAM, Philadelphia (1991), 404–415.

[200] J. Opatrný, D. Sotteau, N. Srinivasan, K. Thulasiraman, DCC linear congruential graphs: a new class of interconnection networks. *IEEE Trans. Comput.* **45** (1996) No. 2, 156–164.

[201] S. Perennes, A proof of Jean de Rumeur's Conjecture *Discrete Appl. Math.* **74** (1997) 295–299.

[202] J. Plesník, One method for proving the impossibility of certain Moore graphs, *Discrete Mathematics* **8** (1974) 363–376.

[203] J. Plesník and Š. Znám, Strongly geodetic directed graphs, *Acta F. R. N. Univ. Comen. - Mathematica* **XXIX** (1974) 29–34.

[204] H. J. Quaife, On $(d, k, \mu)$ graphs, *IEEE Trans. Computers* **C-18** (1969) 270–272.

[205] J. J. Quisquater, Chordal rings, communicated February 27 1998.

[206] C. Roos and A.J. van Zanten, On the existence of certain generalized Moore geometries, Part I, *Discrete Mathematics* **51** (1984) 179–190.

[207] C. Roos and A.J. van Zanten, On the existence of certain generalized Moore geometries, Part II, *Discrete Mathematics* **51** (1984) 277–282.

[208] G. Sabidussi, Graphs without dead ends, *Europ. J. Combinatorics* **17** (1996) 69–87.

[209] M. Sampels and S. Schöf, Massively parallel architectures for parallel discrete event simulation, *Proceedings of the 8th European Simulation Symposium (ESS'96)* **2** (1996) 374–378.

[210] S.B. Seidman, Network structure and minimum diameter, *Social Networks* **5** (1983) No. 3, 269–287.

[211] J. Šiagiová, Minimum diameter lifts of complete graphs, *Proc. Conf. 60th Anniversary of the Fac. Civil Eng.*, Slovak University of Technology, Mathematics and Physics, Bratislava (1998) 229–233.

[212] J. Šiagiová, Composition of regular coverings of graphs, *J. Electrical Engg.* **50** (1999) No. 10, 75–77.

[213] J. Šiagiová, A note on the McKay-Miller-Širáň graphs, *Journal of Combinatorial Theory B* **81** (2001) 205–208.

[214] J. Šiagiová, A Moore-like bound for graphs of diameter 2 and given degree obtained as regular lifts of small graphs, *Acta Math. Univ. Comen.*, to appear.

[215] J. Šiagiová, Composition of regular coverings of graphs and voltage assignments, *Australasian J. Combin.*, to appear.

[216] J. Šiagiová, Permutation voltage assignments and constructions of large $(\Delta, D)$-graphs, in preparation.

[217] J. Šiagiová and R. Simanjuntak, A Moore bound for graphs embedded in surfaces, submitted.

[218] R. Simanjuntak and E.T. Baskoro, The uniqueness of almost Moore digraphs with degree 4 and diameter 2, *Proceeding ITB* **32:1** (2000) 7–11.

[219] R. Simanjuntak and M. Miller, Largest planar digraphs of diameter two, *Proceedings of AWOCA'02*, Frazer Island, Australia, July 2002, 43–52.

[220] Slamin, *Diregularity of Digraphs Close to Moore Bound*, PhD Thesis, Department of Computer Science and Software Engineering, The University of Newcastle (2001).

[221] Slamin, E. T. Baskoro and M. Miller, Diregularity of digraphs of out-degree three and order two less than Moore bound, *Proceedings of AWOCA'01*, Bandung, Indonesia, July 2001, 167–173.

[222] Slamin and M. Miller, Diregularity of digraphs close to Moore bound, *Prosiding Konperensi Nasional Matematika X*, *MIHMI* **6** (2000) No. 5, 185–192.

[223] W.F. Smyth, The construction of large graphs of given maximum degree and diameter, *Congressus Numerantium* **51** (1986) 277–285.

[224] W.F. Smyth, An upper bound on the order of graphs of given maximum degree and

diameter, *Congressus Numerantium* **57** (1987) 201–211.

[225] J. Soares, Maximum diameter of regular digraphs, *Journal of Graph Theory* **16** (1992) No. 5, 437–450.

[226] T. Soneoka, Super edge-connectivity of dense digraphs and graphs, *Discrete Appl. Mathematics* **37/38** (1992) 511–523.

[227] R. Stanton and D. Cowan, Note on a “square” functional equation, *SIAM Review* **12** (1970) 277–279.

[228] R.G. Stanton, S.T.E. Seah and D.D. Cowan, Nonexistence of an extremal graph of a certain type, *J. Austral. Math. Soc.* (Series A) **30** (1980) 55–64.

[229] R.M. Storwick, Improved Construction Techniques for $(d, k)$ Graphs, *IEEE Trans. on Computers* (1970) 1214–1216.

[230] S.A. Tishchenko, The maximum size of a planar graph ($\Delta = 3$, $D = 3$), *Fundam. Prikl, Math.* **7** (2001) No. 1, 159-171.

[231] S.A. Tishchenko, The maximum order of a graph of diameter 2 with a fixed Euler characteristic, *Fundam. Prikl, Math.* **7** (2001) No. 4, 1203-1225.

[232] S. Toueg and K. Steiglitz, The design of small-diameter network by local search, *IEEE Trans. on Computers* **C-28** (1979) 537–542.

[233] M.E. Watkins, Connectivity of transitive graphs, *Journal of Combinatorial Theory* **8** (1970) 23–29.

[234] C.K. Wong, A combinatorial problem related to multimodule memory organization, *J. Ass. Comput. Mach.* **21** (1974) 392–402.

[235] C. K. Wong, D. Coppersmith, A combinatorial problem related to multimodule memory organizations, *J. ACM* **21** (1974) 392–402.

[236] P.K. Wong, Cages - a survey, *J. Graph Theory* **6** (1982) No. 1, 1–22.

[237] J.M. Xu, An inequality relating the order, maximum degree, diameter and connectivity of a strongly connected digraph. *Acta Math. Appl. Sinica (English Ser.)* **8** (1992) No. 2, 144–152.

[238] J.M. Xu and K. Xu, On $k$-diameter of $k$-connected graphs, *Appl. Math. J. Chinese Univ. Ser. B* **16** (2001) No. 3, 231-236.

[239] Y. Yang and J. Lin, Largest planar graphs and largest maximal planar graphs of diameter two, *J. Comput. Appl. Math.* **144** (2002) No. 1-2, 349-358.

[240] J.L.A. Yebra, M.A. Fiol and J. Fábrega, Semiregular bipartite Moore graphs, *Ars Combin.* **16** (1983) 131–139.

[241] A.J. van Zanten, A short proof of a theorem of Bannai and Ito, *Discrete Mathematics* **78** (1989) 321–325.

[242] A. Zlatoš, The diameter of lifted digraphs, *Australasian J. Combin.* **19** (1999) 73–82.

[243] Š. Znám, On existence and regularity of graphs with certain properties, *Discrete Mathematics* **16** (1976) 83–90.