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RENEWAL SERIES AND SQUARE-ROOT BOUNDARIES FOR BESSEL PROCESSES

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Abstract
We show how a description of Brownian exponential functionals as a renewal series gives access to the law of the hitting time of a square-root boundary by a Bessel process. This extends classical results by Breiman and Shepp, concerning Brownian motion, and recovers by different means, extensions for Bessel processes, obtained independently by Delong and Yor.

Let B_t be the standard real valued Brownian motion and for $\nu > 0$, introduce the geometric Brownian motion $\mathscr{E}_t^{(-\nu)}$ and its exponential functional $\mathscr{A}_t^{(-\nu)}$

$$\mathscr{E}_t^{(-\nu)} := exp(B_t - \nu t)$$

$$\mathscr{A}_t^{(-\nu)} := \int_0^t (\mathscr{E}_s^{(-\nu)})^2 ds.$$

Lamperti's representation theorem [5] applied to $\mathscr{E}_t^{(-\nu)}$ states

$$\mathscr{E}_t^{(-\nu)} = R^{(-\nu)}_{\mathscr{A}_t^{(-\nu)}} \tag{0.1}$$

where $(R_u^{(-\nu)}, u \leq T_0(R^{(-\nu)}))$ denotes the Bessel process of index $(-\nu)$ (equivalently of dimension $\delta = 2(1-\nu)$), starting at 1, which is an $\mathbb{R}_+$-valued diffusion with infinitesimal generator $\mathscr{L}^{(-\nu)}$

given by

$$\mathscr{L}^{(-\nu)}f(x)=\frac{1}{2}f''(x)+\frac{1-2\nu}{2x}f'(x),\quad f\in C_b^2(\mathbb{R}_+^\star).$$

Let us remark that, in the special case $\nu=1/2$, equation (0.1) is nothing else but the Dubins-Schwarz representation of the exponential martingale $\mathscr{E}_t^{(-1/2)}$ as Brownian motion time changed with $\mathscr{A}_t^{(-1/2)}$.

For a short summary of relations between Bessel processes and exponentials of Brownian motion, see e.g. Yor [10].

Let us consider now the following random variable Z, which is often called a perpetuity in the mathematical finance literature:

$$Z:=\mathscr{A}_\infty^{(-\nu)}=\int_0^\infty(\mathscr{E}_s^{(-\nu)})^2ds$$

We deduce directly from (0.1) that

$$\mathscr{A}_\infty^{(-\nu)}=T_0(R^{(-\nu)})$$

where $T_0:=\inf\{u:X_u=0\}$, and it is well-known (see [4], [11]), that

$$\mathscr{A}_\infty^{(-\nu)}\overset{(law)}{=}\frac{1}{2\gamma_\nu}\tag{0.2}$$

where γ_ν is a gamma variable with parameter ν (i.e. with density $\frac{1}{\Gamma(\nu)}x^{\nu-1}e^{-x}\mathbf{1}_{\mathbb{R}_+}$).

Our main result characterizes the law of the hitting time of a parabolic boundary by $R_u^{(-\nu)}$ which corresponds to a Bessel process of dimension $d<2$.

Theorem 1. *Let $0<b<c$, and $\sigma:=\inf\{u:(R_u^{(-\nu)})^2=\frac{1}{c}(b+u)\}$ with $R_0^{(-\nu)}=1$.*

$$E[(b+\sigma)^{-s}]=c^{-s}\frac{E[(1+2b\gamma_{\nu+s})^{-s}]}{E[(1+2c\gamma_{\nu+s})^{-s}]},\quad \textit{for any } s\geq 0\tag{0.3}$$

Proof: using the strong Markov property and the stationarity of the increments of Brownian motion, we obtain that for any stopping time τ of the Brownian motion

$$\mathscr{A}_\infty^{(-\nu)}=:Z=\mathscr{A}_\tau^{(-\nu)}+(\mathscr{E}_\tau^{(-\nu)})^2Z'$$

where Z' is independent of $(\mathscr{A}_\tau^{(-\nu)},\mathscr{E}_\tau^{(-\nu)})$ and $Z\overset{(law)}{=}Z'$.

This implies, by (0.1), that Z satisfies the following affine equation (see [8] for many results about these equations)

$$\mathscr{A}_\infty^{(-\nu)}=:Z=\mathscr{A}_\tau^{(-\nu)}+(R^{(-\nu)}_{\mathscr{A}_\tau^{(-\nu)}})^2Z'\tag{0.4}$$

where Z' is independent of $(\mathscr{A}_\tau^{(-\nu)},R^{(-\nu)}_{\mathscr{A}_\tau^{(-\nu)}})$ and $Z\overset{(law)}{=}Z'$.

Obviously, $\sigma<T_0(R^{(-\nu)})$. Taking now :

$$\tau=\inf\{t:(R^{(-\nu)}_{\mathscr{A}_t^{(-\nu)}})^2=\frac{1}{c}(b+\mathscr{A}_t^{(-\nu)})\}$$

we get $\mathscr{A}_\tau^{(-\nu)}=\sigma$, and the identity in law

$$b+Z\overset{(law)}{=}(b+\sigma)(1+\frac{Z}{c})\tag{0.5}$$

where the variables σ and Z on the right-hand side are independent.
As a result, we obtain the Mellin transform of $b+\sigma$ which is:

$$E[(b+\sigma)^{-s}] = c^{-s}\frac{E[(b+Z)^{-s}]}{E[(c+Z)^{-s}]}$$

But, from (0.2)

$$E[(b+\sigma)^{-s}] = c^{-s}\frac{E[(2\gamma_\nu)^s\frac{1}{(1+2b\gamma_\nu)^s}]}{E[(2\gamma_\nu)^s\frac{1}{(1+2c\gamma_\nu)^s}]}$$

which gives the result. □

One can now use the duality between the laws of Bessel processes of dimension d and $4-d$ to get the analogous result of Theorem 1, and recover the result of Delong [2], [3], and Yor [9] which deals with the case $d \geq 2$.

Theorem 2. *Let* $0 < b < c$, *and* $\sigma := \inf\{u : (R_u^{(\nu)})^2 = \frac{1}{c}(b+u)\}$ *with* $R_0^{(\nu)} = 1$.

$$E[(b+\sigma)^{-s}] = c^{-s}\frac{E[(1+2b\gamma_s)^{-s+\nu}]}{E[(1+2c\gamma_s)^{-s+\nu}]}, \quad \textit{for any } s \geq 0. \tag{0.6}$$

Proof : it is based on the following classical relation between the laws of the Bessel processes with indices ν and $-\nu$:

$$\mathscr{P}_x^{(\nu)}{}_{|\mathscr{F}_t} = \frac{(X_{t\wedge T_0})^{2\nu}}{x^{2\nu}}.\mathscr{P}_x^{(-\nu)}{}_{|\mathscr{F}_t} \tag{0.7}$$

which implies that

$$E_1^{(\nu)}[(b+\sigma)^{-s}] = E_1^{(-\nu)}[X_\sigma^{2\nu}(b+\sigma)^{-s}] = \frac{1}{c^\nu}E_1^{(-\nu)}[(b+\sigma)^{-s+\nu}]$$

Theorem 1 gives the result. □

Finally, it is easily shown, thanks to the classical representations of the Whittaker functions (see Lebedev [6] page 279), that the right-hand sides of (0.3) and (0.6) are expressed in terms of ratios of Whittaker functions. Let us recall their integral representation:

$$W_{k,m}(z) = \frac{e^{-z/2}z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k-\frac{1}{2}+m}(1+\frac{t}{z})^{k-\frac{1}{2}+m}e^{-t}dt$$

whenever $\Re(m-k+\frac{1}{2}) \geq 0$ and $\arg(z) < \pi$.
Using this identitity, the rhs of (0.3) and (0.6) take respectively the form

$$c^{-s}\frac{e^{\frac{1}{4b}}W_{\frac{1-\nu}{2}-s,\frac{\nu}{2}}(\frac{1}{2b})}{e^{\frac{1}{4c}}W_{\frac{1-\nu}{2}-s,\frac{\nu}{2}}(\frac{1}{2c})} \quad \text{and} \quad c^{-s}\frac{e^{\frac{1}{4b}}W_{\frac{1+\nu}{2}-s,\frac{\nu}{2}}(\frac{1}{2b})}{e^{\frac{1}{4c}}W_{\frac{1+\nu}{2}-s,\frac{\nu}{2}}(\frac{1}{2c})}.$$

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