An isomorphism theorem for random interlacements

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Abstract

We consider continuous-time random interlacements on a transient weighted graph. We prove an identity in law relating the field of occupation times of random interlacements at level $u$ to the Gaussian free field on the weighted graph. This identity is closely linked to the generalized second Ray-Knight theorem of [2], [4], and uniquely determines the law of occupation times of random interlacements at level $u$.

Keywords: random interlacements; Gaussian free field; isomorphism theorem; generalized second Ray-Knight theorem.

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0 Introduction

In this note we consider continuous-time random interlacements on a transient weighted graph $E$. We prove an identity in law, which relates the field of occupation times of random interlacements at level $u$ to the Gaussian free field on $E$. The identity can be viewed as a kind of generalized second Ray-Knight theorem, see [2], [4], and characterizes the law of the field of occupation times of random interlacements at level $u$. We now describe our results and refer to Section 1 for details. We consider a countable, locally finite, connected graph, with vertex set $E$, endowed with non-negative symmetric weights $c_{x,y} = c_{y,x}$, $x, y \in E$, which are positive exactly when $x, y$ are distinct and $\{x, y\}$ is an edge of the graph. We assume that the induced discrete-time random walk on $E$ is transient. Its transition probability is defined by

$$p_{x,y} = \frac{c_{x,y}}{\lambda_x}, \text{ where } \lambda_x = \sum_{z \in E} c_{x,z}, \text{ for } x, y \in E. \quad (0.1)$$

In essence, continuous-time random interlacements consist of a Poisson point process on a certain space of doubly infinite $E$-valued trajectories marked by their duration at each step, modulo time-shift. A non-negative parameter $u$ plays the role of a multiplicative factor of the intensity of this Poisson point process, which is defined on a suitable canonical space $(\Omega, A, P)$. The field of occupation times of random interlacements at level $u$ is then defined for $x \in E$, $u \geq 0$, $\omega \in \Omega$, by (see (1.8) for the precise expression)

$$L_{x,u}(\omega) = \lambda_x^{-1} \times \text{the total duration spent at } x \text{ by the trajectories modulo time-shift with label at most } u \text{ in the cloud } \omega \quad (0.2)$$

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(informally, the durations of the successive steps of a trajectory are described by independent exponential variables of parameter 1, but occupation times at \( x \) get rescaled by a factor \( \lambda_x^{-1} \)). The Gaussian free field on \( E \) is the other ingredient of our isomorphism theorem. Its canonical law \( P^G \) on \( \mathbb{R}^E \) is such that

\[
\text{under } P^G, \text{ the canonical field } \varphi_x, x \in E, \text{ is a centered Gaussian field with covariance } E^{P^G}[\varphi_x \varphi_y] = g(x, y), \text{ for } x, y \in E, \tag{0.3}
\]

where \( g(\cdot, \cdot) \) stands for the Green function attached to the walk on \( E \), see (1.3). The main result of this note is the next theorem:

**Theorem 0.1.** For each \( u \geq 0 \),

\[
\begin{align*}
\left( L_{x,u} + \frac{1}{2} \varphi_x^2 \right)_{x \in E} & \text{ under } P \otimes P^G, \text{ has the same law as } \\
\left( \frac{1}{2} (\varphi_x + \sqrt{2u})^2 \right)_{x \in E} & \text{ under } P^G. \tag{0.4}
\end{align*}
\]

This theorem provides for each \( u \) an identity in law very much in the spirit of the so-called generalized second Ray-Knight theorems, see Theorem 1.1 of [2] or Theorem 8.2.2 of [4]. Remarkably, although we are in a transient set-up, (0.4) corresponds to the so-called generalized second Ray-Knight theorems, see Theorem 1.1 of [2] or Theorem 8.2.2 of [4].

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We now explain how this note is organized. In Section 1, we provide precise definitions and recall useful facts. Section 2 develops the approximation procedure for \( (L_{x,u})_{x \in E} \). We give two proofs of the main Theorem 2.1, and an extension appears in Remark 2.2. The short Section 3 contains the proof of Theorem 0.1, and a variation of (0.4) in Remark 3.1. In Section 4, we present an application to the study of the large \( u \) behavior of \( (L_{x,u})_{x \in E} \), see Theorem 4.1.

**1 Notation and useful results**

In this section we provide additional notation and recall some definitions and useful facts related to random walks, potential theory, and continuous-time interlacements. We consider the spaces \( \tilde{W}_+ \) and \( \tilde{W} \) of infinite, and doubly infinite, \( E \times (0, \infty) \)-valued sequences, such that the \( E \)-valued sequences form an infinite, respectively doubly-infinite, nearest-neighbor trajectory spending finite time in any finite subset of \( E \), and such that the \( (0, \infty) \)-valued components have an infinite sum in the case of \( \tilde{W}_+ \), and infinite “forward” and “backward” sums, when restricted to positive and negative indices, in the case of \( \tilde{W} \). We write \( Z_n, \sigma_n \), with \( n \geq 0 \), or \( n \in \mathbb{Z} \), for the respective \( E \)- and \( (0, \infty) \)-valued coordinates on \( \tilde{W}_+ \) and \( \tilde{W} \). We denote by \( P_x, x \in E \), the law on \( \tilde{W}_+ \), endowed with its canonical \( \sigma \)-algebra, under which \( Z_{n}, n \geq 0 \), is distributed as simple random walk starting at \( x \), and \( \sigma_{n}, n \geq 0 \), are i.i.d. exponential variables with parameter 1, independent from the \( Z_n, n \geq 0 \). We denote by \( E_x \) the corresponding expectation. Further, when \( \rho \) is
a measure on $E$, we write $P_x$ for the measure $\sum_{x \in E} \rho(x) P_x$, and $E_x$ for the corresponding expectation. We denote by $X_t$, $t \geq 0$, the continuous-time random walk on $E$, with constant jump rate 1, defined for $t \geq 0$, $\hat{w} \in \hat{W}_+$, by

$$X_t(\hat{w}) = Z_k(\hat{w}), \text{ when } \sigma_0(\hat{w}) + \cdots + \sigma_{k-1}(\hat{w}) \leq t < \sigma_0(\hat{w}) + \cdots + \sigma_k(\hat{w}) \quad (1.1)$$

(by convention the term bounding $t$ from below vanishes when $k = 0$). Given $U \subseteq E$, we write $H_U = \inf\{t \geq 0; X_t \in U\}$, $\hat{H}_U = \inf\{t > 0; X_t \in U\}$, and for some $s \in (0, t)$, $X_s \neq X_0$, and $T_U = \inf\{t \geq 0; X_t \notin U\}$, for the entrance time in $U$, the hitting time of $U$, and the exit time from $U$. We denote by $g_U(\cdot, \cdot)$ the Green function of the walk killed when exiting $U$

$$g_U(x, y) = \frac{1}{N_y} E_x \left[ \int_0^{T_U} 1\{X_s = y\} ds \right], \text{ for } x, y \in E. \quad (1.2)$$

The function $g_U(\cdot, \cdot)$ is known to be symmetric and finite (due to the transience assumption we have made). When $U = E$, no killing takes place (i.e. $T_U = \infty$), and we simply write

$$g(x, y) = g_{U=E}(x, y), \text{ for } x, y \in E, \quad (1.3)$$

for the Green function. Given a finite subset $K$ of $U$, the equilibrium measure and capacity of $K$ relative to $U$ are defined by

$$e_{K,U}(x) = P_x[\hat{H}_K > T_U] \lambda_x 1_K(x), \text{ for } x \in E, \quad (1.4)$$

$$\text{cap}_{U}(K) = \sum_{x \in E} e_{K,U}(x). \quad (1.5)$$

When $U = E$, we simply drop $U$ from the notation, and refer to $e_K$ and $\text{cap}(K)$, as the equilibrium measure and the capacity of $K$. Further, the probability to enter $K$ before exiting $U$ can be expressed as

$$P_x[H_K < T_U] = \sum_{y \in E} g_U(x, y) e_{K,U}(y), \text{ for } x \in E. \quad (1.6)$$

We now turn to the description of continuous-time random interlacements on the transient weighted graph $E$. We write $\hat{W}_+$ for the space $\hat{W}$ (introduced at the beginning of this section), modulo time-shift, i.e. $\hat{W}^* = W/\sim$, where for $\hat{w}, \hat{w}' \in \hat{W}$, $\hat{w} \sim \hat{w}'$ means that $\hat{w}(\cdot) = \hat{w}'(\cdot + k)$ for some $k \in \mathbb{Z}$. We denote by $\pi^* : \hat{W} \to \hat{W}^*$ the canonical map, and endow $\hat{W}^*$ with the $\sigma$-algebra consisting of sets with inverse image under $\pi^*$ belonging to the canonical $\sigma$-algebra of $\hat{W}$. The continuous-time interlacement point process is a Poisson point process on the space $\hat{W}^* \times \mathbb{R}_+$. Its intensity measure has the form $\nu(\hat{w}\pi^* d\nu, \hat{w})$, where $\nu$ is the $\sigma$-finite measure on $\hat{W}^*$ such that for any finite subset $K$ of $E$, the restriction of $\nu$ to the subset of $\hat{W}^*$ consisting of those $\hat{w}$ for which the $E$-valued trajectory modulo time-shift enters $K$, is equal to $\pi^* \circ \hat{Q}_K$, the image of $\hat{Q}_K$ under $\pi^*$, where $\hat{Q}_K$ is the finite measure on $\hat{W}$ specified by

i) $\hat{Q}_K(Z_0 = x) = e_K(x), \text{ for } x \in E,$

ii) when $e_K(x) > 0$, conditionally on $Z_0 = x$, $(Z_n)_{n \geq 0}$, $(Z_{-n})_{n \geq 0}$, $(\sigma_n)_{n \in \mathbb{Z}}$ are independent, respectively distributed as simple random walk starting at $x$, as simple random walk starting at $x$ conditioned never to return to $K$, and as a doubly infinite sequence of i.i.d. exponential variables with parameter 1.

(1.7)
As in [6], the canonical continuous-time random interlacement point process is then constructed similarly to (1.16) of [5], or (2.10) of [8], on a space \((\Omega, \mathcal{A}, \mathbb{P})\), with \(\omega = \sum_{i \geq 0} \delta(\bar{\omega}^i, u_i)\) denoting a generic element of \(\Omega\). A central object of interest in this note is the random field of occupation times of random interlacements at level \(u \geq 0\):

\[
L_{x,u}(\omega) = \frac{1}{\lambda_x} \sum_{i \geq 0} \sum_{n \in \mathbb{Z}} \sigma_n(\bar{w}_i) 1\{Z_n(\bar{w}_i) = x, u_i \leq u\}, \quad \text{for } x \in E, \omega \in \Omega,
\]

where \(\omega = \sum_{i \geq 0} \delta(\bar{\omega}^i, u_i)\) and \(\pi^*(\bar{w}_i) = \bar{w}_i^\ast\), for each \(i \geq 0\).

The Laplace transform of \((L_{x,u})_{x \in E}\) has been computed in [6]. More precisely, given a function \(f: E \to \mathbb{R}\), such that \(\sum_{y \in E} g(x,y)\langle f(y) \rangle < \infty\), for \(x \in E\), one sets

\[
Gf(x) = \sum_{y \in E} g(x,y)\langle f(y) \rangle, \quad \text{for } x \in E.
\]

One knows from Theorem 2.1 and Remark 2.4 of [6], that when \(V: E \to \mathbb{R}_+\) has finite support and

\[
sup_{x \in E} GV(x) < 1,
\]

one has the identity

\[
E\left[\exp\left\{-\sum_{x \in E} V(x)L_{x,u}\right\}\right] = \exp\{-u\langle V, (I + GV)^{-1}\rangle\}, \quad \text{for } u \geq 0,
\]

where the notation \(\langle f, g \rangle\) stands for \(\sum_{x \in E} f(x)g(x)\), when \(f, g\) are functions on \(E\) such that the previous sum converges absolutely, and \(1_E\) denotes the constant function identically equal to 1 on \(E\).

## 2 An approximation scheme for random interlacements

In this section we develop an approximation scheme for \((L_{x,u})_{x \in E}\) in terms of the fields of local times of certain finite state space Markov chains. The main result is Theorem 2.1, but Remark 2.2 states a by-product of the approximation scheme concerning the random interlacement at level \(u\). This has a similar flavor to Theorem 4.17 of [7], where one gives one of several possible meanings to random interlacements viewed as “Markovian loops going through infinity”, see also Le Jan [3], p. 85. We consider a non-decreasing sequence \(U_n, n \geq 1\), of finite connected subsets of \(E\), increasing to \(E\), as well as \(x_\ast\) some fixed point not belonging to \(E\). We introduce the sets \(E_n = U_n \cup \{x_\ast\}\), for \(n \geq 1\), and endow \(E_n\) with the weights \(c^n_{x,y}\), \(x,y \in E_n\), obtained by “collapsing \(U_n\) on \(x_\ast\)”, that is, for any \(n \geq 1\), and \(x,y \in U_n\), we set

\[
\begin{align*}
c^n_{x,y} &= \begin{cases} c_{x,y}, & \text{if } x,y \not\in U_n, \\ 0, & \text{otherwise.} \end{cases} \\
c^n_{x,y} &= \sum_{z \in E \setminus U_n} c_{z,y},
\end{align*}
\]

and otherwise set \(c^n_{x,y} = 0\) (i.e. \(c^n_{x_\ast,x_\ast} = 0\)). We also write

\[
\lambda^n_x = \sum_{y \in E_n} c^n_{x,y}, \quad \text{for } x \in E_n \quad \text{(in particular } \lambda^n_x = \lambda_x, \text{ when } x \in U_n). \quad \text{(2.2)}
\]

We tacitly view \(U_n\) as a subset of both \(E\) and \(E_n\). We consider the canonical simple random walk in continuous time on \(E_n\), attached to the weights \(c^n_{x,y}\), \(x,y \in E_n\), with jump rate equal to 1. We write \(X^n_t, t \geq 0\), for its canonical process, \(P^n_{x}\) for its canonical law starting from \(x \in E_n\), and \(E^n_x\) for the corresponding expectation. The local time of
this Markov chain is defined by
\begin{equation}
\ell_t^{n,x} = \frac{1}{\lambda} \int_0^t 1\{X^n_s = x\} \, ds, \quad \text{for } x \in E_n \text{ and } t \geq 0.
\end{equation}

The function \( t \geq 0 \mapsto \ell_t^{n,x} \geq 0 \) is continuous, non-decreasing, starts at 0, and \( P^n_y \)-a.s. tends to infinity, as \( t \) goes to infinity (the walk on \( E_n \) is irreducible and recurrent). By convention, when \( x \in E \setminus U_n \), we set \( \ell_t^{n,x} = 0 \), for all \( t \geq 0 \). We introduce the right-continuous inverse of \( \ell_t^{n,x} \).

\[ \tau_u^n = \inf\{ t \geq 0; \ell_t^{n,x} > u \}, \text{ for any } u \geq 0. \]  

We are now ready for the main result of this section. We tacitly endow \( R^E \) with the product topology, and convergence in distribution, as stated below (and in the sequel), corresponds to convergence in law of all finite dimensional marginals.

**Theorem 2.1.** \((u \geq 0)\)

\((\ell_t^{n,x}, \tau_u^n)_{x \in E} \) under \( P^n_x \) converges in distribution to \((Lx,u)_{x \in E} \) under \( P \).  

**Proof.** We give two proofs. First proof: We denote by \( T \) the set of piecewise-constant, right-continuous, \( E \cup \{x_\ast\} \)-valued trajectories, which at a finite time reach \( x_\ast \), and from that time onwards remain equal to \( x_\ast \). We endow \( T \) with its canonical \( \sigma \)-algebra. Under \( P^n_x \), one has almost surely two infinite sequences \( R_\ell, \ell \geq 1 \) and \( D_\ell, \ell \geq 1 \),

\[ R_1 = 0 < R_2 < \cdots < R_\ell < D_\ell < \cdots \]  

of successive returns \( R_\ell \) of \( X^n \) to \( x_\ast \), and departures \( D_\ell \) from \( x_\ast \), which tend to infinity. One introduces the random point measure on \( T \)

\[ \Gamma_u^n = \sum_{\ell \geq 1} 1\{D_\ell < \tau_u^n\} \delta_{(X^n_{D_\ell+1},X^n_{D_\ell}) \leq \tau_u^n-D_\ell}, \quad u \geq 0, \]  

which collects the successive excursions of \( X^n \) (out of \( x_\ast \) until first return to \( x_\ast \)) that start before \( \tau_u^n \). By classical Markov chain excursion theory we know that \( \Gamma_u^n \) is a Poisson point measure on \( T \) with intensity measure

\[ \gamma_u^n(\cdot) = u P^n_x[(X^n_{\tau_u^n})_{s \geq 0} \in \cdot] \text{ on } T, \]  

where \( T_{U_n} \) stands for the exit time of \( X^n \) from \( U_n \) and \( \kappa_n \) for the measure on \( U_n \)

\[ \kappa_n(y) = \lambda^n \sum_{x,y \in U_n} c_{x,y}^{n} = c_{x,y}^{n}, \quad \text{for } y \in U_n. \]  

When starting in \( U_n \), the Markov chains \( X \) on \( E \), and \( X^n \) on \( E_n \), have the same evolution strictly before the exit time of \( U_n \). Denoting by \( (X)_0 \leq \cdot \leq T_{U_n} \) the random element of \( T \), which equals \( X_s \), for \( 0 \leq s \leq T_{U_n} \) and \( x_\ast \) for \( s \geq T_{U_n} \), we see that

\[ \gamma_u^n(\cdot) = u P_{\kappa_n}[(X)_0 \leq \cdot \leq T_{U_n} \in \cdot], \text{ for all } u \geq 1, u \geq 0. \]  

Let \( K \) be a finite subset of \( E \), and assume \( n \) large enough so that \( K \subseteq U_n \). We introduce the point measure on \( T \) obtained by selecting the excursions in the support of \( \Gamma_u^n \) that enter \( K \), and only keeping track of their trajectory after they enter \( K \), that is

\[ \mu_{K,u}^n = \theta_{H_K} \circ (1\{H_K < \infty\}) \Gamma_u^n, \]  

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where $\theta_t, t \geq 0$, stands for the canonical shift on $\mathcal{T}$, and we use similar notation on $\mathcal{T}$ as below (1.1). By (2.8), (2.10) it follows that

\[
\begin{align*}
\mu^n_{K,u} & \text{ is a Poisson point measure on } \mathcal{T} \text{ with intensity measure} \\
\gamma^n_{K,u}(\cdot) &= u \rho^n_K([X_0 \leq t-u_n]) \text{ on } \mathcal{T},
\end{align*}
\] (2.12)

where $\rho^n_K$ is the measure supported by $K$ such that

\[
\rho^n_K(x) = P_{\mu^n_{K,u}}[H_K < T_{u_n}, X_{H_K} = x] = e_{K,U_n}(x), \text{ for } x \in K,
\] (2.13)

where the last equality follows from (1.60) in Proposition 1.8 of [7]. Note that $e_{K,U_n}$ and $e_K$ are concentrated on $K$, and for $x \in K$,

\[
e_{K,U_n}(x) \stackrel{(1.4)}{=} P_x[\tilde{H}_K > T_{u_n}] \lambda_x \xrightarrow{n \to \infty} P_x[\tilde{H}_K = \infty] = e_K(x).
\] (2.14)

Consider $V : E \to \mathbb{R}_+$ supported in $K$, and $\Phi : T \to \mathbb{R}_+$, the map

\[
\Phi(w) = \sum_{x \in E} V(x) \frac{1}{X_x} \int_0^{\infty} 1{\{w(s) = x\}} ds, \text{ for } w \in \mathcal{T}.
\]

The measure $\mu^n_{K,u}$ contains in its support the pieces of the trajectory $X^n$ up to time $\tau^n_u$, where $X^n$ visits $K$, see (2.11), and we have

\[
E^n_x \left[ \exp \left\{ -\sum_{x \in E} V(x) \ell^n_{x,u} \right\} \right] = E^n_x \left[ \exp \left\{ -\langle \mu^n_{K,u}, \Phi \rangle \right\} \right] \stackrel{(2.12)}{=} \exp \left\{ \int_T (e^{-\Phi - 1} d\gamma^n_{K,u}) \right\} = \exp \left\{ u E_{e_{K,U_n}} \left[ e^{-\int_0^{T_{u_n}} \frac{V(x)}{X_x} ds} - 1 \right] \right\} \xrightarrow{n \to \infty} E \left[ \exp \left\{ -\sum_{x \in E} V(x) L_{x,u} \right\} \right],
\] (2.15)

where we used (2.14) and the fact that $T_{u_n} \uparrow \infty$, $P_y$-a.s., for $x \in E$, for the limit in the last line, and a similar calculation as in (2.5) of [6] for the last equality. Since $K$ and the function $V : E \to \mathbb{R}_+$, supported in $K$, are arbitrary, the claim (2.5) follows.

**Second Proof:** We will now make direct use of (1.11). The argument is more computational, but also of interest. We consider $K$ and $V$ as above, as well as a positive number $\lambda$. We assume $n$ large enough so that $K \subseteq U_n$. We further make a smallness assumption on the non-negative function $V$ (supported in $K$):

\[
\sup_{x \in E} (GV)(x) + \lambda^{-1} \sum_{x \in K} V(x) < 1.
\] (2.16)

We define the operator $G_n$ on $\mathbb{R}^{E_n}$ attached to the kernel $g_n(\cdot, \cdot)$ in a similar fashion to (1.9), where we use the notation

\[
g_n(x, y) = g_{U_n}(x, y) + \lambda^{-1}, \text{ for } x, y \in E_n,
\] (2.17)

and we have set $g_{U_n}(x,, \cdot) = g_{U_n}(\cdot, x) = 0$, by convention, to define $g_{U_n}(\cdot, \cdot)$ on $E_n \times E_n$. Since $g_{U_n}(\cdot, \cdot) \leq g(\cdot, \cdot)$ on $E \times E$, it follows from (2.16) that $\sup_{x \in E_n} (G_n V)(x) < 1$, where we have set $V(x, \cdot) = 0$, by convention, so that the operator $I + G_n V$ is invertible.

We introduce the positive number

\[
a_n = \int_0^{\infty} \lambda e^{-\lambda u} E^{n}_{x,u} \left[ e^{-\sum_{x \in E} V(x) \ell^n_{x,u}} \right] du,
\] (2.18)

where we recall that $\ell^{n,x}_t = 0$, when $x \in E \setminus U_n$. Using (2.93), (2.41), (2.71) of [7], or by (8.44) and Remark 3.10.3 of Marcus-Rosen [4], we know that

\[
a_n = (I + G_n V)^{-1} 1_{E_n}(x_\ast).
\] (2.19)
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We then define the function $h_n$ on $E_n$ and the real number $b_n$:

$$h_n = (I + G_n V)^{-1} 1_{E_n} \quad \text{and} \quad b_n = \sum_{x \in K} V(x) h_n(x). \quad (2.20)$$

We let $G_{U_n}$ be the operator on $\mathbb{R}^{E_n}$ attached to the kernel $g_{U_n}(\cdot, \cdot)$ on $E_n \times E_n$, in a similar fashion to (1.9). By (2.17) and (2.20), we have

$$h_n + G_{U_n} h_n + \lambda^{-1} b_n 1_{E_n} = 1_{E_n}, \quad \text{so that} \quad h_n = \left(1 - \frac{b_n}{\lambda}\right)(1 + G_{U_n} V)^{-1} 1_{E_n}, \quad (2.21)$$

noting that the above inverse is well defined by the same argument used below (2.17). By the second equality in (2.20) it follows that

$$b_n = \left(1 - \frac{b_n}{\lambda}\right) \sum_{x \in K} V(x)(I + G_{U_n} V)^{-1}(x) = \left(1 - \frac{b_n}{\lambda}\right)(V, (I + G_{U_n} V)^{-1} 1_{E_n}), \quad (2.22)$$

where we refer to below (1.11) for notation, $G_{U_n}$ is the operator on $\mathbb{R}^E$ attached to the kernel $g_{U_n}(\cdot, \cdot)$ on $E \times E$, and the last equality follows by writing the Neumann series for $(I + G_{U_n} V)^{-1}$ and $(I + G_{U_n} V)^{-1}$ (note that $V \geq 0$ and (2.16) straightforwardly imply the convergence of these series in the respective operator norms induced by $L^\infty(E_n)$ and $L^\infty(E)$). We can now solve for $b_n$. Noting that $a_n = h_n(x) = 1 - \frac{b_n}{\lambda}$, by (2.21), we find

$$a_n = (1 + \lambda^{-1}(V, (I + G_{U_n} V)^{-1} 1_{E_n}))^{-1}. \quad (2.23)$$

Using the Neumann series for $(I + G_{U_n} V)^{-1}$, and applying dominated convergence together with the fact that $g_{U_n}(\cdot, \cdot) \uparrow g(\cdot, \cdot)$ on $E \times E$, we see that

$$a_n \xrightarrow{n \to \infty} (1 + \lambda^{-1}(V, (I + GV)^{-1} 1_{E_n}))^{-1}. \quad (2.24)$$

Taking the identity (1.11) into account, we have shown that under (2.16),

$$\lim_n \int_0^\infty \lambda e^{-\lambda u} E^n_{x_n} \left[ e^{-\sum_{x \in K} V(x) E^n_{x,u}} \right] du = \int_0^\infty \lambda e^{-\lambda u} E \left[ e^{-\sum_{x \in K} V(x) E_{x,u}} \right] du. \quad (2.25)$$

Note that when $V : E \to \mathbb{R}_+$ is supported in $K$ and $\sup_{x \in E} GV(x) < 1$, then (2.16) holds for $\lambda$ large (depending on $V$). The expectation under the integral in the left-hand side of (2.25) is non-increasing in $u$, whereas the expectation under the integral in the right-hand side of (2.25) is continuous in $u$ by (1.11). It then follows from [1], p. 193-194, that for $V$ as above,

$$\lim_n E^n_{x_n} \left[ e^{-\sum_{x \in K} V(x) E^n_{x,u}} \right] = E \left[ e^{-\sum_{x \in K} V(x) E_{x,u}} \right], \quad \text{for } u \geq 0. \quad (2.26)$$

This readily implies the tightness of the laws of $((E_{x_n}^n)_{x \in K})$ under $P^n_x$, and uniquely determines the Laplace transform of their possible limit points, see Theorem 6.6.5 of [1]. Letting $K$ vary, the claim (2.5) follows.

**Remark 2.2.** The approximation scheme introduced in this section can also be used to approximate the random interlacement at level $u$, as we now explain. We let $I^n_u$ stand for the trace left on $U_n$ by the walk on $E_n$ up to time $\tau^n_u$:

$$I^n_u = \{ x \in U_n : \tau^n_u > 0 \}. \quad (2.27)$$

By (2.12), (2.14), it follows that for any finite subset $K$ of $E$ and $u \geq 0$,

$$P^n_x[I^n_u \cap K = \emptyset] = P^n_x[I^n_{K,u} = 0] = e^{-u \text{cap}_{u,K}(u)} \left(1 + \frac{1}{n}\right) e^{-u \text{cap}(K)} = P[I^u \cap K = \emptyset]. \quad (2.28)$$
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where $I^u$ stands for the random interlacement at level $u$, that is, the trace on $E$ of doubly infinite trajectories modulo time-shift in the Poisson cloud $\omega$ with label at most $u$. By an inclusion-exclusion argument, see for instance Remark 4.15 of [7] or Remark 2.2 of [5], it follows that, as $n \to \infty$,

$$T^u_n \text{ under } P^*_{x, n}, \text{ converges in distribution to } I^u \text{ under } P, \text{ for any } u \geq 0, \quad (2.29)$$

where the above distributions are viewed as laws on $\{0, 1\}^E$ endowed with the product topology. □

3 Proof of the isomorphism theorem

In this short section we combine Theorem 2.1 and the generalized second Ray-Knight theorem of [2] to prove Theorem 0.1. We also state a variation of (0.4) in Remark 3.1.

Proof of Theorem 0.1: For $U \subseteq G$ we denote by $P_{G, U}^G$ the law on $\mathbb{R}^E$ of the centered Gaussian field with covariance $E_{G, U}[\varphi_x \varphi_y] = g_U(x, y)$, $x, y \in E$ (in particular $\varphi_x = 0$, $P_{G, U}$, a.s., when $x \in E \setminus U$). It follows from the generalized second Ray-Knight theorem, see Theorem 8.2.2 of [4], or Theorem 2.17 of [7], that for $n \geq 1$, $u \geq 0$, in the notation of Section 2,

$$\left(\ell_{n, x}^{n, x} + \frac{1}{2} \varphi_x^2\right)_{x \in U_n} \text{ under } P_{x, n}^n \otimes P_{G, U_n}^G, \text{ has the same law as}$$

$$\left(\frac{1}{2} (\varphi_x + \sqrt{2u})^2\right)_{x \in U_n} \text{ under } P_{G, U_n}^G. \quad (3.1)$$

Since $g_U(\cdot, \cdot) \uparrow g(\cdot, \cdot)$, we see that $P_{G, U_n}^G$ converges weakly to $P^G$ (looking for instance at characteristic functions of finite dimensional marginals). Taking Theorem 2.1 into account we thus see letting $n$ tend to infinity that

$$\left(L_{x, u} + \frac{1}{2} \varphi_x^2\right)_{x \in E} \text{ under } P \otimes P^G, \text{ has the same law as}$$

$$\left(\frac{1}{2} (\varphi_x + \sqrt{2u})^2\right)_{x \in E} \text{ under } P^G, \quad (3.2)$$

and Theorem 0.1 is proved. □

Remark 3.1. Let us mention a variation on (0.4) of Theorem 0.1. By Theorem 1.1 of [2], one knows that for $u \geq 0$, $a \in \mathbb{R}$, $n \geq 1$,

$$\left(\ell_{n, x}^{n, x} + \frac{1}{2} (\varphi_x + a)^2\right)_{x \in U_n} \text{ under } P_{x, n}^n \otimes P_{G, U_n}^G, \text{ has the same law as}$$

$$\left(\frac{1}{2} (\varphi_x + \sqrt{2u + a^2})^2\right)_{x \in U_n} \text{ under } P_{G, U_n}^G. \quad (3.3)$$

Letting $n$ tend to infinity, the same argument as above shows that for $u \geq 0$, and $a \in \mathbb{R}$,

$$\left(L_{x, u} + \frac{1}{2} (\varphi_x + a)^2\right)_{x \in E} \text{ under } P \otimes P^G, \text{ has the same law as}$$

$$\left(\frac{1}{2} (\varphi_x + \sqrt{2u + a^2})^2\right)_{x \in E} \text{ under } P^G. \quad (3.4)$$

□

4 An application

We illustrate the use of Theorem 0.1 and show how one can study the large $u$ asymptotics of $(L_{x, u})_{x \in E}$ and in particular recover Theorem 5.1 of [6], see also Remark 5.2 of [6]. We denote by $x_0$ some fixed point of $E$. 

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**Theorem 4.1.** As $u \to \infty$,

$$
\left( \frac{1}{u} L_{x,u} \right)_{x \in E} \text{ converges in distribution to the constant field equal to 1,}
$$

(4.1)

$$
\left( \frac{L_{x,u} - u}{\sqrt{2u}} \right)_{x \in E} \text{ converges in distribution to } (\varphi_x)_{x \in E} \text{ under } P^G.
$$

(4.2)

In particular, as $u \to \infty$,

$$
\left( \frac{L_{x,u} - L_{x,0}}{\sqrt{2u}} \right)_{x \in E} \text{ converges in distribution to } (\varphi_x - \varphi_{x_0})_{x \in E} \text{ under } P^G.
$$

(4.3)

**Proof.** We first prove (4.1). To this end we note that $P^G$-a.s., for $x \in E$,

$$
\frac{1}{2u} \varphi_x^2 \to 0 \text{ and } \frac{1}{2u} (\varphi_x + \sqrt{2u})^2 \to 1, \text{ as } u \to \infty.
$$

(4.4)

Thus Theorem 0.1 implies that $\frac{1}{u} L_{x,u}$ converges in distribution to the constant 1 as $u$ tends to infinity, and (4.1) follows. We then observe that (4.3) is a direct consequence of (4.2), and turn to the proof of (4.3). Note that by Theorem 0.1

$$
\left( \frac{L_{x,u} - u}{\sqrt{2u}} + \frac{1}{2u} \varphi_x^2 \right)_{x \in E} \text{ under } P \otimes P^G, \text{ has the same law as }
$$

(4.5)

$$
\left( \frac{1}{2\sqrt{2u}} [2\varphi_x + \sqrt{2u})^2 - 2u] \right)_{x \in E}.
$$

Note also that for each $x \in E$, $P^G$-a.s., as $u \to \infty$,

$$
\frac{1}{2\sqrt{2u}} \varphi_x^2 \to 0, \text{ and }
$$

(4.6)

$$
\frac{1}{2\sqrt{2u}} [2\varphi_x + \sqrt{2u})^2 - 2u] = \frac{1}{2\sqrt{2u}} \varphi_x^2 + \varphi_x \to \varphi_x.
$$

(4.7)

Looking at the characteristic function of finite dimensional marginals of the fields in the first and second line of (4.5), we readily obtain (4.3).

**Remark 4.2.** In view of the above illustration of the use of Theorem 0.1, one can naturally wonder about the nature of its scope as a transfer mechanism between random interlacements and the Gaussian free field.

**References**


