On the Number of Minimum Neighbourhood Sets in Paths and Cycles

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Abstract

This paper is concerned with total number of minimum neighbourhood sets in paths and cycles. We establish the relation between minimum number of neighbourhood sets in paths and cycles and the blocks of partial balanced incomplete block design with m-association scheme.

Keywords: Minimum neighbourhood sets, neighbourhood number and PBIBD with m-association scheme.

1 Introduction

By a graph we mean a finite undirected graph without loops or multiple lines. For a graph $G$, let $V(G)$ and $E(G)$ respectively denote the point set and the line set of graph $G$.

For $v \in V$, the closed neighbourhood of $V$ is $N[v] = \{u \in v : uv \in E(G)\} \bigcup \{v\}$. A subset $S$ of $V(G)$ is a neighbourhood set of $G$ if $G = \bigcup_{v \in S} (N(v))$. Where $(N(v))$ is the subgraph of $G$ induced by $N[v]$. The neighbourhood number $\eta(G)$ of $G$ is the minimum cardinality of a neighbourhood set of $G$. This number was introduced by Sampathkumar and Neeralagi in the year 1985.

We now determine the total number of minimum neighbourhood sets in paths
and cycles. We denote \( \eta_M(G) \) as the total number of minimum neighbourhood sets (mn-sets) in a graph \( G \).

We also consider the set of all minimum neighbourhood sets in cycles as the blocks of some partial balanced incomplete block designs with \( m \)-association scheme. Any undefined terms and notation, reader may refer to F. Harary[5] and for more details about neighbourhood set see [3, 4]. We refer the reader to see [1, 2] for more details about PBIBD and dominating set.

2 Minimum Neighbourhood Sets in Paths and Cycles

We denote \( \eta_M(G) \) as the total number of minimum neighbourhood sets in a graph \( G \). Here we find \( \eta_M(P_n) \) by characterizing the path \( P_n \) into two classes \( n = 2^k \) and \( 2^k + 1 \) for any integer \( k \geq 0 \). Similarly we characterize the \( C_n \) into two classes \( n = 2^k \) and \( 2^k + 1 \), for any integer \( k \geq 2 \).

2.1 Basic Results

Observation 2.1. In \( P_n \) with the labellings \( v_1, v_2, v_3, \ldots v_{n-2}, v_{n-1}, v_n \), one of the following holds:

(i) \( v_1, v_{n-1} \in S \)
(ii) \( v_2, v_n \in S \)
(iii) \( v_2, v_{n-1} \in S \),

where \( S \) is any arbitrary minimum neighbourhood sets in \( P_n \). From here onwards, we write mn-sets for minimum neighbourhood sets.

The following theorem is straightforward from the observation 1.

Theorem 2.2. \( \eta_M(P_{2k+1}) = 1 \) for \( k \geq 0 \) We prove the above theorem straightforward from the observation 1.

Lemma 2.3. There is only one minimum neighbourhood set containing \( v_1 \) and \( v_{2k+1} \) in \( P_{2k+2} \) with the labellings \( v_1, v_2, \ldots v_{2k}, v_{2k+1}, v_{2k+2} \).

Proof. Clearly \( S = \{v_1, v_3, \ldots v_{2k-1}, v_{2k+1}\} \) is an mn-set in \( P_{2k+2} \) containing \( v_1 \) and \( v_{2k+1} \) which proves the existence.

To prove the uniqueness, consider \( S - \{v_1, v_{2k+1}\} \) which is a neighbourhood set in \( P_{2k-2} \) with the labellings \( \{v_3, v_4, \ldots v_{2k-2}, v_{2k-1}\} \). The set \( S - \{v_1, v_{2k+1}\} \) is not only a neighbourhood set, but also minimum.

Since \( |S - \{v_1, v_{2k+1}\}| = (k + 1) - 2 = \eta(P_{2k-1}) \)

By Theorem 1, \( S - \{v_1, v_{2k+1}\} \) is unique mn-set in \( P_{2k-1} \), therefore \( S \) is unique in \( P_{2k+2} \).
Lemma 2.4. There is only one mn-set containing $v_2$ and $v_{2k+2}$ in $P_{2k+2}$ with the labellings $v_1, v_2, \ldots, v_{2k+2}$.

Proof. The proof is similar to that of Lemma 2, by counting mn-sets from the right of the labellings of $P_{2k+2}$ as was done from the right of the labellings of $P_{2k+2}$ as was done from the left in Lemma 2.

Lemma 2.5. There are exactly $k$ mn-sets containing $v_2$ and $v_{2k+1}$ in $P_{2k+2}$ with labellings $v_1, v_2, \ldots, v_{2k+1}, v_{2k+2}$.

Proof. For $k = 1$, the set $\{v_2, v_3\}$ is an mn-set in $P_{2(1)+2} = P_4$ and $k = 2$, the sets are $\{v_2, v_4, v_5\}$ and $\{v_2, v_3, v_5\}$ are mn-sets in $P_{2(2)+2} = P_6$ as stated in lemma. Thus the result is true for $k = 1$ and $k = 2$.

Assume the result is true for $k - 1$, that there are exactly $k - 1$ mn-sets in $P_{2(k-1)+2} = P_{2k}$ containing $v_2$ and $v_{2k-1}$.

Let $S_1, S_2, \ldots, S_{k-1}$ be mn-sets containing $v_2$ and $v_{2k-1}$. Now consider $P_{2k+2}$ with the labellings $v_1, v_2, \ldots, v_{2k+1}, v_{2k+2}$. Clearly $S_1 \cup \{v_{2k+1}\}$ is an mn-set containing $v_2$ and $v_{2k-1}$ in $P_{2k+2}$.

Since $|S_1 \cup \{v_{2k+1}\}| = k + 2\quad \eta(P_{2k+2})$ for $i = 1, 2, \ldots, (k-1)$.

Lemma 2.6. There are exactly $(k-1)$ mn-sets containing $v_2$ and $v_{2k-1}$ in $P_{2k}$ with labellings $v_1, v_2, \ldots, v_{2k-1}, v_{2k}$.

Proof. We prove the lemma by induction on $K$. For $k = 2$ the set $\{v_2, v_3\}$ is an mn-set in $P_{2(2)} = P_4$. For $k = 3$, the set $\{v_2, v_4, v_5\}$ and $\{v_2, v_3, v_5\}$ are mn-sets in $P_{2(2)} = P_6$.

Thus the result is true for $k = 2$ and $k = 3$. Assume that the result is true for $k - 2$, that is there are exactly $k - 2$ mn-sets in $P_{2(k-2)} = P_{2k-4}$ containing $v_2$ and $v_{2k-3}$ in $P_{2k-4}$. Let $A_1, A_2, \ldots, A_{k-2}$ be mn-sets containing $v_2$ and $v_{2k-3}$ in $P_{2k-4}$ with the labellings $v_1, v_2, \ldots, v_{2k-1}, v_{2k}$.

Clearly $A_i \cup \{v_{2k-1}\}$ is an mn-set containing $v_2$ and $v_{2k-1}$ in $P_{2k}$. Since $|A_i \cup \{v_{2k-1}\}| = k\quad = \eta(P_{2k})$, for $i = 1, 2, 3, \ldots, (k-2)$.

Hence the result is true for $(k-1)$.

Lemma 2.7. There is no mn-set containing $v_1$ and $v_{2k+2}$ in $P_{2k+2}$.

Proof. On the contrary, assume $D$ be a mn-set containing $v_1$ and $v_{2k+2}$ in $P_{2k+2}$. Then $D = \{v_1, v_{2k+2}\}$ is a neighbourhood of $P_{2k}$ with the labellings $v_3, v_4, v_5, \ldots, v_{2k}$. Since $v_2$ and $v_{2k+1}$ are neighbourhoods of $v_1$ and $v_{2k+2}$ in $P_{2k+2}$.

Thus,

$K = \eta(P_{2k}) \leq |D - \{v_1, v_{2k+2}\}|$

$= k + 1 - 2$

$= k - 1$ a contradiction.
Lemma 2.8. There is no mn-set containing $v_1$ and $v_{2k+1}$ in $P_{2k+1}$.

Proof. On contrary, assume that $S$ be a mn-set containing $v_1$ and $v_{2k+1}$ in $P_{2k+1}$. Then, $S = \{v_1, v_{2k+1}\}$ is a neighbourhood set in $P_{2k-1}$ with labellings $v_2, v_3, \ldots, v_{2k}$, since $v_2$ and $v_{2k}$ are neighbourhoods of $v_1$ and $v_{2k+1}$ of mn-sets $S$ in $P_{2k+1}$.

Thus,

$$K = \eta(P_{2k+1}) \leq |S - \{v_1, v_{2k+1}\}|$$

$$= k - 2,$$

a contradiction.

2.2 Main Results

Lemma 2.9. There is no mn-set containing $v_1$ and $v_{2k}$ in $P_{2k}$.

Proof. The proof is similar to the above Lemma 5.

Theorem 2.10. For any integer $k \geq 0$,

$$\eta(P_{2k+4}) = \eta_M(P_{2k+2}) + 1$$

Proof. Let us label the vertices of $P_{2k+4}$ as $v_1, v_2, \ldots, v_{2k+3}, v_{2k+4}$. Let $S$ be the mn-set containing $v_1$ and $v_{2k+1}$ in $P_{2k+4}$ which is the only one mn-set follows from Lemma 2.

Therefore $S_1 = S \cup \{v_{2k+3}\}$

$$= \eta(P_{2k+2}) + 1$$

$$= (k + 2) + 1$$

$$= k + 3$$

$$= \eta P_{2k+4}.$$  

Let $S'$ be the mn-set containing $v_2$ and $v_{2k+1}$ in $P_{2k+2},$ which is the only one mn-set. Similarly follows, from the Lemma 3, therefore

$S_2 = S' \cup v_{2k+4}$ is the mn-set for $P_{2k+4},$ since

$$|S_2| = |S' \cup v_{2k+4}|$$

$$= |S'| + 1$$

$$= \eta(P_{2k+2}) + 1$$

$$= k + 3$$

$$= \eta(P_{2k+4})$$

Again by the above Lemma let $S_1, S_2, \ldots, S_k$ be the mn-set containing $v_1$ and $v_{2k+1}$ in $P_{2k+4}$.

For each set $S_i$ containing $v_1$ and $v_{2k+1}$ in $P_{2k+4}$, there is an mn-set $S'_i = S_1 \cup \{v_{2k+3}\}$ in $P_{2k+4}$. Hence by the Lemma(2,3,4) and Observation 1 the number of mn-sets are

$$\eta_M(P_{2k+2}) = k + 2$$

$$\eta_M(P_{2k+4}) = k + 3$$

$$= (k + 2) + 1$$

$$= \eta_M(P_{2k+2}) + 1.$$
Theorem 2.11. $\eta_M(C_{2k}) = 2$ for $k \geq 1$.

Example 2.12. $k = 4, \eta_M(C_8) = 2$

![Figure 1](image)

The minimum neighbourhood sets are $\{1, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$.

Lemma 2.13. There is one and only one minimum neighbourhood set containing two adjacent vertices in $C_{2k+3}$.

**Proof.** Let us label the vertices of $C_{2k+3}$ as $v_1, v_2, ..., v_{2k-1}, v_{2k}, v_{2k+1}, v_{2k+2}, v_{2k+3}$. Clearly the set $\eta = v_1, v_2, v_4, ..., v_{2k-1}, v_{2k}, v_{2k+2}$ is an mn-set in $C_{2k+3}$ containing two adjacent vertices for example $v_1$ and $v_2$ which proves the existence. Also $D$ is unique. Since $D - \{v_1, v_2\}$ is mn-set in $P_{2k-2}$ with the labeling $v_3, v_4, v_5, ..., v_{2k}$. But by previous theorem, $D - \{v_1, v_2\}$ is uniqueness. $P_{2k-2}$ and hence $D$ is unique in $C_{2k+3}$.

3 Minimum Neighbourhood Sets and PBIBD, Association Scheme Related To Cycle Graphs

The relation between graphs and PBIBD have been studied by many authors. For example H.B.Walikar [2], has been studied the relation between the minimum dominating sets and PBIBD, Anwar Alwardi [1] has been studied the relation between the dominating sets and PBIBD with association scheme for special case of graphs called strongly regular graphs. This motivated us to study the general case for cycles and for the other graphs still open for research. In our paper we denote the parameters of PBIBD in the following order $(v, b, r, k, \lambda_1, \lambda_i)$.

**Definition 3.1.** Given $v$ objects a relation satisfying the following conditions is said to be an association scheme with $m$ classes:

(i) any two objects are either first associates, or second associates, ..., or $m^{th}$ associates, the relation of association being symmetric.
(ii) each object $\alpha$ has $n_i$ $i^{th}$ associates, the number $n_i$ being independent of $\alpha$.

(iii) if two objects $\alpha$ and $\beta$ are $i^{th}$ associates, then the number of objects which are $j^{th}$ associates of $\alpha$ and $k^{th}$ associates of $\beta$ is $p_{jk}^i$ and is independent of the pair of $i^{th}$ associates $\alpha$ and $\beta$. Also $p_{jk}^i=p_{kj}^i$.

If we have association scheme for the $v$ objects we can define a PBIBD as the following definition.

**Definition 3.2.** The PBIB design is arrangement of $v$ objects into $b$ sets (called blocks) of size $k$ where $k < v$ such that

(i) every object is contained in exactly $r$ blocks.

(ii) each block contains $k$ distinct objects.

(iii) Any two objects which are $i^{th}$ associates occur together in exactly $\lambda_i$ blocks.

**Proposition 3.3.** From the neighbourhood sets of $C_5$ we get the PBIBD with parameters $(5,5,3,3,1,2)$ with two association scheme.

Proof: The cycle $C_5$ with the labelling $1, 2, 3, 4, 5$ as in Figure 2 whose mn-sets are $(1,3,5)$, $(1,2,4)$, $(2,3,5)$, $(1,3,4)$ and $(2,4,5)$ and these are the blocks of the PBIBD with parameters $(5,5,3,3,1,2)$ with two association scheme defined as following:

Any two vertices say $\alpha$ and $\beta$, $\alpha$ is first associates to $\beta$ if $\alpha$ and $\beta$ appear together once in the minimum neighbourhood sets, and $\alpha$ is second associates to $\beta$ if $\alpha$ and $\beta$ appear together twice in the minimum neighbourhood sets we give the table of association scheme as following:

<table>
<thead>
<tr>
<th>Elements</th>
<th>First Associates</th>
<th>Second Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,5</td>
<td>3,4</td>
</tr>
<tr>
<td>2</td>
<td>1,3</td>
<td>4,5</td>
</tr>
<tr>
<td>3</td>
<td>2,4</td>
<td>1,5</td>
</tr>
<tr>
<td>4</td>
<td>3,5</td>
<td>1,2</td>
</tr>
<tr>
<td>5</td>
<td>1,4</td>
<td>2,3</td>
</tr>
</tbody>
</table>

and $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. 
Proposition 3.4. From the neighbourhood sets of $C_7$ we get the PBIBD with parameters $(7,7,4,4,1,2,3)$ with three association scheme.

Proof: The cycle $C_7$ with the labelling $1, 2, 3, 4, 5, 6, 7$ as in Figure 3 whose mn-sets are $(1,3,5,6)$, $(1,2,4,6)$, $(2,3,5,7)$, $(1,3,4,6)$, $(2,4,5,7)$, $(2,4,6,7)$ and $(1,3,5,7)$ and these are the blocks of the PBIBD with parameters $(7,7,4,4,1,2,3)$ with three association scheme defined as following:

Any two vertices say $\alpha$ and $\beta$, $\alpha$ is first associates to $\beta$ if $\alpha$ and $\beta$ appear together once in the minimum neighbourhood sets, and $\alpha$ is second associates to $\beta$ if $\alpha$ and $\beta$ appear together twice in the minimum neighbourhood sets and $\alpha$ is third associates to $\beta$ if $\alpha$ and $\beta$ appear together thrice in the minimum neighbourhood sets we give the table of association scheme as following:

<table>
<thead>
<tr>
<th>Elements</th>
<th>First Associates</th>
<th>Second Associates</th>
<th>Third Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,7</td>
<td>4,5</td>
<td>3,6</td>
</tr>
<tr>
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<td>5,6</td>
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<tr>
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<td>2,4</td>
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<td>1,5</td>
</tr>
<tr>
<td>4</td>
<td>3,5</td>
<td>1,7</td>
<td>2,6</td>
</tr>
<tr>
<td>5</td>
<td>4,6</td>
<td>1,2</td>
<td>3,7</td>
</tr>
<tr>
<td>6</td>
<td>5,7</td>
<td>2,3</td>
<td>1,4</td>
</tr>
<tr>
<td>7</td>
<td>1,6</td>
<td>3,4</td>
<td>2,5</td>
</tr>
</tbody>
</table>

and $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Figure 2

Figure 3
Proposition 3.5. From the neighbourhood sets of $C_9$ we get the PBIBD with parameters $(9,9,5,5,1,2,3,4)$ with four association scheme.

Proof: The cycle $C_9$ with the labelling $1,2,3,4,5,6,7,8,9$ as in Figure 4 whose mn-sets are $(1,3,5,6,8), (1,2,4,6,8), (2,3,5,7,9), (1,3,4,6,8), (2,4,5,7,9), (2,4,6,7,9), (1,3,5,7,9), (1,3,5,7,8),$ and $(2,4,6,8,9)$ and these are the blocks of the PBIBD with parameters $(9,9,5,5,1,2,3,4)$ with four association scheme defined as following:

Any two vertices say $\alpha$ and $\beta$, $\alpha$ is first associates to $\beta$ if $\alpha$ and $\beta$ appear together once in the minimum neighbourhood sets, and $\alpha$ is second associates to $\beta$ if $\alpha$ and $\beta$ appear together twice in the minimum neighbourhood sets and $\alpha$ is third associates to $\beta$ if $\alpha$ and $\beta$ appear together thrice in the minimum neighbourhood sets and $\alpha$ is fourth associates to $\beta$ if $\alpha$ and $\beta$ appear together four times in the minimum neighbourhood sets we give the table of association scheme as following:

<table>
<thead>
<tr>
<th>Elements</th>
<th>First Associates</th>
<th>Second Associates</th>
<th>Third Associates</th>
<th>Fourth Associates</th>
</tr>
</thead>
<tbody>
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<td>5,6</td>
<td>3,8</td>
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<td>6,7</td>
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<td>3,5</td>
<td>1,7</td>
<td>8,9</td>
<td>2,6</td>
</tr>
<tr>
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<td>4,6</td>
<td>2,8</td>
<td>1,9</td>
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<td>6</td>
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<td>8</td>
<td>7,9</td>
<td>2,5</td>
<td>3,4</td>
<td>1,6</td>
</tr>
<tr>
<td>9</td>
<td>1,8</td>
<td>3,6</td>
<td>4,5</td>
<td>2,7</td>
</tr>
</tbody>
</table>

and

\[
P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}
\]

and

\[
P_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]
Proposition 3.6. From the neighbourhood sets of $C_{11}$ we get the PBIBD with parameters $(11,11,6,6,1,2,3,4,5)$ with five association scheme.

Proof: The cycle $C_{11}$ with the labelling $1,2,3,4,5,6,7,8,9,10,11$ as in Figure 5 whose mn-sets are $(1,3,5,6,8,10), (1,2,4,6,8,10), (2,3,5,7,9,11), (1,3,4,6,8,10), (2,4,5,7,9,11), (2,4,6,7,9,11), (1,3,5,7,9,10), (1,3,5,7,8,10), (2,4,6,8,9,11), (1,3,5,7,9,11)$ and $(2,4,6,8,10,11)$ and these are the blocks of the PBIBD with parameters $(11,11,6,6,1,2,3,4,5)$ with five association scheme defined as following:

Any two vertices say $\alpha$ and $\beta$, $\alpha$ is first associates to $\beta$ if $\alpha$ and $\beta$ appear together once in the minimum neighbourhood sets, $\alpha$ is second associates to $\beta$ if $\alpha$ and $\beta$ appear together twice in the minimum neighbourhood sets, $\alpha$ is third associates to $\beta$ if $\alpha$ and $\beta$ appear together thrice in the minimum neighbourhood sets, $\alpha$ is fourth associates to $\beta$ if $\alpha$ and $\beta$ appear together four times in the minimum neighbourhood sets and $\alpha$ is fifth associates to $\beta$ if $\alpha$ and $\beta$ appear together five times in the minimum neighbourhood sets, we give the table of association scheme as following:

<table>
<thead>
<tr>
<th>Elements</th>
<th>$1^{st}$ Associates</th>
<th>$2^{nd}$ Associates</th>
<th>$3^{rd}$ Associates</th>
<th>$4^{th}$ Associates</th>
<th>$5^{th}$ Associates</th>
</tr>
</thead>
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<td>1,11</td>
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<td>1,2</td>
<td>3,11</td>
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</tr>
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<td>5,11</td>
<td>2,3</td>
<td>1,4</td>
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<tr>
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<td>8,10</td>
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<td>3,4</td>
<td>2,5</td>
<td>7,11</td>
</tr>
<tr>
<td>10</td>
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<td>4,5</td>
<td>3,6</td>
<td>1,8</td>
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<td>11</td>
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<td>3,8</td>
<td>5,6</td>
<td>4,7</td>
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</tr>
</tbody>
</table>

and
\[ P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \]

From the previous examples we can deduce that for any cycle graph \( C_{2k+1} \), where \( k \geq 2 \), we can define PBIBD from the minimum neighbourhood set with \( 2k + 1 \) points and also \( 2k + 1 \) blocks because as we have proved previously. Also it is clear that the size of any block is the neighbourhood number of \( C_{2k+1} \) and association scheme with \( \lfloor (2k + 1)/2 \rfloor \).

**References**


