We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the operator generated.

1. Introduction

In the rectangle $\Omega = (0,1) \times (0,T)$, we consider the equation

$$f(x,t) = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right)$$

(1.1)

with the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1),$$

(1.2)

the final condition

$$\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1),$$

(1.3)

the Dirichlet condition

$$u(0,t) = 0 \quad \forall \ t \in (0,T),$$

(1.4)

and the integral condition

$$\int_{l}^{1} u(x,t) dx = 0, \quad 0 \leq l < 1, \quad t \in (0,T).$$

(1.5)
In addition, we assume that the function \( a(x,t) \) and its derivatives satisfy the conditions

\[
0 < a_0 < a(x,t) < a_1 \quad \forall x, t \in \Omega,
\]

\[
\left| \frac{\partial a}{\partial x} \right| \leq b \quad \forall x, t \in \Omega,
\]

\[
c'_k < \left| \frac{\partial^k u}{\partial t^k} (x,t) \right| < c_k \quad \forall x, t \in \Omega, k = 1, 3,
\]

with \( c'_1 > 0 \).

Over the last few years, many physical phenomena were formulated into nonlocal mathematical models with integral boundary conditions [1, 9, 10, 11]. The reader should refer to [13, 14] and the references therein. The importance of these kinds of problems has also been pointed out by Samarskii [22]. This type of boundary value problems has been investigated in [2, 3, 4, 6, 7, 8, 12, 18, 19, 20, 23, 25] for parabolic equations, in [21, 24] for hyperbolic equations, and in [15, 16, 17] for mixed-type equations. The basic tool in [5, 15, 16, 17, 20, 25] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation.

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of the problem (1.1)–(1.5). For this, we consider the solution of problem (1.1)–(1.5) as a solution of the operator equation

\[
Lu = \mathcal{F},
\]

where the operator \( L \) has domain of definition \( D(L) \) consisting of functions \( u \in L^2(\Omega) \) such that \( (\partial^{k+1} u / \partial t^k \partial x)(x,t) \in L^2(\Omega), k = 1, 3 \) and satisfying the conditions (1.4)-(1.5).

The operator \( L \) is considered from \( E \) to \( F \), where \( E \) is the Banach space consisting of function \( u \in L^2(\Omega) \), with the finite norm

\[
\| u \|^2_E = \int_\Omega \Theta(x) \left[ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx \, dt
\]

\[
+ \int_\Omega \Theta(x) \left[ \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \right] dx \, dt
\]

\[
+ \int_\Omega \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx \, dt.
\]

\( F \) is the Hilbert space of functions \( \mathcal{F} = (f,0,0,0), f \in L^2(\Omega) \), with the finite norm

\[
\| \mathcal{F} \|^2_F = \int_\Omega \Theta(x) |f(x,t)|^2 dx \, dt,
\]
where

\[ \Theta(x) = \begin{cases} 
(1 - l)^2, & 0 < x \leq l, \\
(1 - x)^2, & l \leq x < 1,
\end{cases} \]

\[ \Phi(x) = \begin{cases} 
0, & 0 < x < l, \\
1, & l \leq x < 1.
\end{cases} \]

(2.4)

3. An energy inequality and its application

**Theorem 3.1.** For any function \( u \in D(L) \), the a priori estimate

\[ \| u \|_E \leq k \| Lu \|_F \quad \text{for} \ u \in D(L), \] \hspace{1cm} (3.1)

where \( k^2 = 40 \exp(cT)/k_1 \) with \( k_1 = \inf \{ 1/4, (c'_3 - 3cc'_1 + 3c^2c'_1 - c^3a_1 - b^2)/2, a_0^2/2, (3/2)(ca_0 - c_1) \} \). The constant \( c \) satisfies

\[ \sup_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} \right) < c < \inf_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} + 1 \right), \] \hspace{1cm} (3.2)

\[ c'_3 - 3cc'_1 + 3c^2c'_1 - c^3a_1 - b^2 > 0, \]

\[ c'_2 - 2cc'_1 + c^2a_1 + ca_0 - c_1 > 0. \]

**Proof.** Let

\[ Mu = \begin{cases} 
(1 - l)^2 \frac{\partial^3 u}{\partial t^3}, & 0 < x < l, \\
(1 - x)^2 \frac{\partial^3 u}{\partial t^3} + 2(1 - x)J_x \frac{\partial^3 u}{\partial t^3}, & l < x < 1,
\end{cases} \]

(3.3)

where \( J_x u = \int_x^1 u(x,t) \, dx \).

We consider the quadratic form obtained by multiplying (1.1) by \( \exp(-ct)\overline{Mu} \), with the constant \( c \) satisfying (3.2), integrating over \( \Omega = (0,1) \times (0,T) \), and taking the real part:

\[ \Phi(u,u) = \text{Re} \int_\Omega \exp(-ct) f(x,t) \overline{Mu} \, dx \, dt. \] \hspace{1cm} (3.4)
By substituting the expression of \( Mu \) in (3.4), integrating with respect to \( x \), and using the Dirichlet and integral conditions, we obtain

\[
\text{Re} \int_\Omega \exp(-ct) f(x,t) Mu dx \, dt
\]

\[
= \int_0^T \int_0^1 \Theta(x) \exp(-ct) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx \, dt
\]

\[
- \frac{3}{2} \int_0^T \int_0^1 \Theta(x) \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx \, dt
\]

\[
+ \int_0^T \int_0^1 \Theta(x) \left( \frac{\partial a}{\partial t} \right)^2 - 3c \left( \frac{\partial^2 a}{\partial t^2} \right)^2 + 3c \left( \frac{\partial a}{\partial t} \right)^2 - c^3 a \left| \frac{\partial u}{\partial x} \right|^2 dx \, dt
\]

\[
+ \int_0^T \int_0^1 \exp(-ct) \left| J_s \left( \frac{\partial^3 u}{\partial t^3} \right) \right|^2 dx \, dt
\]

\[
- 2 \text{Re} \int_0^T \int_0^1 \exp(-ct) a(x,t) u \left( \frac{\partial^3 u}{\partial t^3} \right) dx \, dt
\]

\[
+ \int_0^T \Theta(x) \exp(-ct) a(x,t) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx |_{t=T}
\]

\[
- \int_0^T \Theta(x) \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left| \frac{\partial u}{\partial x} \right|^2 dx |_{t=T}
\]

\[
- \frac{1}{2} \int_0^T \Theta(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} - 2c \left( \frac{\partial a}{\partial t} \right) + c^2 a \right] \left| \frac{\partial u}{\partial x} \right|^2 dx |_{t=T}
\]

\[
- 2 \text{Re} \int_0^T \int_0^1 \exp(-ct) \frac{\partial a}{\partial x} u \left( \frac{\partial^3 u}{\partial t^3} \right) dx \, dt.
\]

Integrating by parts \(-2 \text{Re} \int_0^T \int_0^1 \exp(-ct) a(x,t) u \left( \frac{\partial^3 u}{\partial t^3} \right) dx \, dt\) with respect to \( t \), and using the initial conditions, the final conditions, and the elementary inequalities, we obtain

\[
\int_0^T \int_0^1 \Theta(x) \exp(-ct) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx \, dt
\]

\[
- \frac{3}{2} \int_0^T \int_0^1 \Theta(x) \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx \, dt
\]

\[
+ \int_0^T \int_0^1 \Theta(x) \left( \frac{\partial a}{\partial t} \right)^2 - 3c \left( \frac{\partial^2 a}{\partial t^2} \right)^2 + 3c \left( \frac{\partial a}{\partial t} \right)^2 - c^3 a \left| \frac{\partial u}{\partial x} \right|^2 dx \, dt
\]

\[
+ \int_0^T \int_0^1 \exp(-ct) \left| J_s \left( \frac{\partial^3 u}{\partial t^3} \right) \right|^2 dx \, dt
\]

\[
- \frac{3}{2} \int_0^T \int_0^1 \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial u}{\partial t} \right|^2 dx \, dt
\]

\[
+ \int_0^1 \Theta(x) \exp(-ct) \left[ a - \left( \frac{\partial a}{\partial t} - ca \right) \right] \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx |_{t=T}
\]
\[- \int_0^1 \Theta(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \left| \frac{\partial a}{\partial t} - ca \right| \right] \left| \frac{\partial u}{\partial x} \right|^2 \, dx |_{t=T} \]
\[+ \int_0^1 \Phi(x) \exp(-ct) \left[ a - \left| \frac{\partial a}{\partial t} - ca \right| \right] \left| \frac{\partial u}{\partial t} \right|^2 \, dx |_{t=T} \]
\[- \int_0^1 \Phi(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \left| \frac{\partial a}{\partial t} - ca \right| \right] |u|^2 \, dx |_{t=T} \]
\[\leq 17 \int_0^T \int_1^0 \Theta(x) \exp(-ct) |f|^2 \, dx \, dt. \]  

(3.6)

From (1.1), we get
\[\int_\Omega \Theta(x) a^2 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \, dt \]
\[\leq 2 \int_\Omega \Theta(x) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt + 2 \int_\Omega \Theta(x) \left( \frac{\partial a}{\partial x} \right)^2 \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt \]
\[+ 4 \int_\Omega \Theta(x) |f|^2 \, dx \, dt. \]  

(3.7)

Combining this last inequality with (3.6) and using the conditions (3.2) yield
\[\int_\Omega \Theta(x) \left[ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] \, dx \, dt \]
\[+ \int_\Omega \Theta(x) \left[ \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \right] \, dx \, dt + \int_\Omega \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] \, dx \, dt \]  

(3.8)

which is the desired inequality. \hfill \Box

It can be proved in a standard way that the operator \( L : E \to F \) is closable. Let \( \overline{L} \) be the closure of this operator, with the domain of definition \( D(\overline{L}) \).

**Definition 3.2.** A solution of the operator equation \( \overline{L} u = \overline{f} \) is called a strong solution of problem (1.1)–(1.5).

The a priori estimate (3.1) can be extended to strong solutions, that is, we have the estimate
\[\| u \|_E \leq c \| \overline{L} u \|_F \quad \forall u \in D(\overline{L}). \]  

(3.9)

This last inequality implies the following corollaries.

**Corollary 3.3.** A strong solution of (1.1)–(1.5) is unique and depends continuously on \( \overline{f} \).

**Corollary 3.4.** The range \( R(\overline{L}) \) of \( \overline{L} \) is closed in \( F \) and \( R(\overline{L}) = R(L) \).
Corollary 3.4 shows that to prove that problem (1.1)–(1.5) has a strong solution for arbitrary $\mathcal{F}$, it suffices to prove that set $R(L)$ is dense in $F$.

4. Solvability of problem (1.1)–(1.5)

To prove the solvability of problem (1.1)–(1.5) it is sufficient to show that $R(L)$ is dense in $F$. The proof is based on the following lemma.

**Lemma 4.1.** Suppose that the function $a(x, t)$ and its derivatives are bounded. Let $u \in D_0(L) = \{u \in D(L), u(x, 0) = 0, (\partial u/\partial t)(x, 0) = 0, (\partial^2 u/\partial t^2)(x, T) = 0\}$. If for $u \in D_0(L)$ and some functions $w(x, t) \in L^2(\Omega)$,

$$
\int_\Omega h(x)f w\,dx\,dt = 0,
$$

where

$$
h(x) = \begin{cases}
1 - l, & 0 < x < l, \\
1 - x, & l < x < 1,
\end{cases}
$$

holds, for arbitrary $u \in D_0(L)$, and then $w = 0$.

**Proof.** The equality (4.1) can be written as follows:

$$
\int_\Omega h(x)\frac{\partial^3 u}{\partial t^3}w\,dx\,dt = \int_\Omega A(t)u\,\tilde{v}\,dx\,dt,
$$

for a given $w(x, t)$, where

$$
v = \begin{cases}
(1 - l)w, & 0 < x < l, \\
 w - \int_l^x \frac{w}{1 - \zeta}d\zeta, & l < x < 1,
\end{cases}
$$

$$
A(t)u = \frac{\partial}{\partial x}\left(h(x)a(x, t)\frac{\partial u}{\partial x}\right),
$$

$$
Nv = \begin{cases}
(1 - l)v, & 0 < x < l, \\
 (1 - x)v + J_x v, & l < x < 1.
\end{cases}
$$

For $v = w - \int_l^x (w/(1 - \zeta))d\zeta$, $l < x < 1$ we deduce

$$
\int_l^x v(\zeta, t)d\zeta = (1 - x)\int_l^x (w/(1 - \zeta))d\zeta,
$$

then

$$
\int_j^1 v(\zeta, t)d\zeta = 0.
$$

Following [25], we introduce the smoothing operators with respect to $t$, $(J_{\epsilon}^{-1}) = (I - \epsilon(\partial^3/\partial t^3))^{-1}$, and $(J_{\epsilon}^{-1})^* = (I + \epsilon(\partial^3/\partial t^3))^{-1}$ which provide the solution of the respective problems:

$$
\begin{align*}
u_{\epsilon} - \epsilon \frac{\partial^3 u_{\epsilon}}{\partial t^3} &= u, & \partial u_{\epsilon}(x, 0) = 0, & \partial^2 u_{\epsilon}(x, 0) = 0, & \frac{\partial^2 u_{\epsilon}}{\partial t^2}(x, T) = 0, \\
v_{\epsilon}^* + \epsilon \frac{\partial^3 v_{\epsilon}^*}{\partial t^3} &= v, & \partial v_{\epsilon}^*(x, 0) = 0, & \partial^2 v_{\epsilon}^*(x, T) = 0.
\end{align*}
$$

(4.5)
And also, we have the following properties: for any \( u \in L^2(0,T) \), the function \( J_{\epsilon}^{-1}u \in W^1_2(0,T) \), \((J_{\epsilon}^{-1})^*u \in W^3_2(0,T)\). If \( u \in D(L)\), \((J_{\epsilon}^{-1})^*u \in D(L)\).

\[
\lim_{\epsilon \to 0} ||J_{\epsilon}^{-1}u - u||_{L^2(0,T)} = 0, \\
\lim_{\epsilon \to 0} ||(J_{\epsilon}^{-1})^*u - u||_{L^2(0,T)} = 0. \tag{4.6}
\]

Substituting the function \( u \) in (4.3) by the smoothing function \( u_{\epsilon} \) and using the relation \( A(t)u_{\epsilon} = J_{\epsilon}^{-1}A(t)u + \epsilon J_{\epsilon}^{-1}B_{\epsilon}(t)u \), where \( B_{\epsilon}(t) = (3\partial/\partial t)((\partial A(t)/\partial t)(\partial u_{\epsilon}/\partial t)) + (\partial^3 A(t)/\partial t^3)u_{\epsilon} \), we obtain

\[
\int_{\Omega} u N \frac{\partial^3 v_{\epsilon}}{\partial t^3} \, dx \, dt = \int_{\Omega} A(t)uv_{\epsilon} \, dx \, dt - \epsilon \int_{\Omega} B_{\epsilon}(t)uv_{\epsilon} \, dx \, dt. \tag{4.7}
\]

The operator \( A(t) \) has a continuous inverse in \( L^2(0,1) \) defined by

\[
A^{-1}(t)g = \begin{cases} 
- \frac{1}{1-l} \int_{l}^{x} \frac{d\zeta}{a(\zeta,t)} g(\eta) \, d\eta + \frac{C_1(t)}{1-l} \int_{0}^{x} \frac{d\zeta}{a(\zeta,t)}, & 0 < x < l, \\
\int_{l}^{x} \frac{-d\zeta}{(1-\zeta)a(\zeta,t)} g(\eta) \, d\eta + C_2(t) \int_{l}^{x} \frac{d\zeta}{(1-\zeta)a(\zeta,t)} + u(l), & l < x < 1,
\end{cases} \tag{4.8}
\]

where

\[
C_1(t) = \frac{(1-l)u(l) + \int_{l}^{1} (d\zeta/a(\zeta,t)) \int_{0}^{x} g(\eta) \, d\eta}{\int_{l}^{1} (d\zeta/a(\zeta,t))},
\]

\[
C_2(t) = \frac{-(1-l)u(l) + \int_{l}^{1} (d\zeta/a(\zeta,t)) \int_{l}^{x} g(\eta) \, d\eta}{\int_{l}^{1} (d\zeta/a(\zeta,t))}. \tag{4.9}
\]

Then we have \( \int_{l}^{x} A^{-1}(t)u = 0 \), hence, the function \( J_{\epsilon}^{-1}u = u_{\epsilon} \) can be represented in the form

\[
u_{\epsilon} = J_{\epsilon}^{-1}A^{-1}(t)A(t)u. \tag{4.10}\]

The adjoint of \( B_{\epsilon}(t) \) has the form

\[
B_{\epsilon}^*(t)v = \frac{1}{a} (J_{\epsilon}^{-1})^* \frac{\partial^3 a}{\partial t^3} \frac{\partial v}{\partial t} + \frac{3}{a} (J_{\epsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial v}{\partial t} \right) - G_{\epsilon}(v)(x) \tag{4.11}
\]

\[
+ \int_{0}^{x} \frac{(d\zeta/a(\zeta,t))}{(d\zeta/a(\zeta,t))} G_{\epsilon}(v)(1),
\]

where

\[
G_{\epsilon}(v)(x) = \int_{0}^{x} \left[ \frac{3}{a} (J_{\epsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial v}{\partial t} \right) + \frac{3}{a^2} \frac{\partial a}{\partial \zeta} (J_{\epsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial v}{\partial t} \right) \right] \, d\zeta. \tag{4.12}
\]
Consequently, equality (4.7) becomes

\[ \int_{\Omega} uN \frac{\partial^3 v^*_\epsilon}{\partial t^3} \, dx \, dt = \int_{\Omega} A(t)u\overline{h}_\epsilon \, dx \, dt, \]  

(4.13)

where \( h_\epsilon = v^*_\epsilon - \epsilon B^*_\epsilon(t)v^*_\epsilon \).

The left-hand side of (4.13) is a continuous linear functional of \( u \), hence the function \( h_\epsilon \) has the derivatives \( \partial h_\epsilon / \partial x \), \( (1-x)(\partial h_\epsilon / \partial x) \in L^2(\Omega) \), and the condition \( h_\epsilon(0,t) = 0 \) is satisfied.

From the equality

\[ (1-x)\frac{\partial h_\epsilon}{\partial x} = \left[ I - \epsilon \frac{1}{a}(J^{-1}_\epsilon)^* \left( \frac{\partial^3 a}{\partial t^3} \right) \right] (1-x)\frac{\partial v^*_\epsilon}{\partial x} - 3\epsilon \frac{1}{a}(J^{-1}_\epsilon)^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} (1-x)\frac{\partial v^*_\epsilon}{\partial x} \right), \]

(4.14)

and since the operator \((J^{-1}_\epsilon)^*\) is bounded in \( L^2(\Omega) \), for sufficiently small \( \epsilon \), we have \( \| \epsilon (1/a)(J^{-1}_\epsilon)^*(\partial^3 a/\partial t^3) \| < 1 \). Hence, the operator \( I - \epsilon (1/a)(J^{-1}_\epsilon)^*(\partial^3 a/\partial t^3) \) has a bounded inverse in \( L^2(\Omega) \). We conclude that \((1-x)(\partial v^*_\epsilon / \partial x) \in L^2(\Omega) \). Similarly, we conclude that \((\partial/\partial x)((1-x)(\partial v^*_\epsilon / \partial x)) \) exists and belongs to \( L^2(\Omega) \), and the condition \( v^*_\epsilon(0,t) = 0 \) is satisfied.

Putting \( u = \int_0^T \int_0^T \int_\eta^T \exp(\epsilon t) v^*_\epsilon \, d\tau \, d\eta \, d\zeta \) in (4.3), where the constant \( \epsilon \) satisfies (3.2) and using the properties of smoothing operator, we obtain

\[ \int_{\Omega} \exp(\epsilon t)v^*_\epsilon \overline{N}v \, dx \, dt = -\int_{\Omega} A(t)u\overline{v}_\epsilon \, dx \, dt - \epsilon \int_{\Omega} A(t)u\frac{\partial^3 v^*_\epsilon}{\partial t^3} \, dx \, dt, \]

(4.15)

and from

\[ -\epsilon \int_{\Omega} A(t)u\frac{\partial^3 v^*_\epsilon}{\partial t^3} \, dx \, dt \]

\[ = 3 \int_{\Omega} h(x) \exp(-ct) \left| \frac{\partial^3 a}{\partial t^2} \right| \left| \frac{\partial^3 u}{\partial t^2 \partial x} \right|^2 \, dx \, dt \]

\[ - 3 \int_{\Omega} h(x) \exp(-ct) \left[ \frac{\partial^3 a}{\partial t^3} - \epsilon \frac{\partial^3 a}{\partial t^2} \right] \left| \frac{\partial^3 u}{\partial t^2 \partial x} \right|^2 \, dx \, dt \]

\[ + 3 \int_0^1 \frac{h(x)}{2} \exp(-ct) \left| \frac{\partial a}{\partial t} \right| \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \bigg|_{t=T} \]

\[ + 3 \int_0^1 \frac{h(x)}{2} \exp(-ct) \left| \frac{\partial^2 a}{\partial t^2} - \epsilon \frac{\partial a}{\partial t} \right| \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \bigg|_{t=T} \]

\[ - \int_{\Omega} h(x) \exp(-ct) a \left| \frac{\partial^3 v^*_\epsilon}{\partial t^3} \right|^2 \, dx \, dt \]

\[ - \int_{\Omega} h(x) \exp(-ct) \frac{\partial^3 a}{\partial t^3} \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial t^2 \partial x} \, dx \, dt, \]

(4.16)
we have
\begin{align}
- \epsilon \text{Re} \int \limits_\Omega A(t) u \frac{\partial^3 v^*}{\partial t^3} dx dt \\
\leq \epsilon \left\{ 3 \int \limits_\Omega h(x) \exp(-ct) \left[ \frac{\partial^2 a}{\partial t^2} + \frac{1}{2} \frac{\partial^2 a}{\partial t^2} - c \frac{\partial^2 a}{\partial t^2} \right] \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
+ \frac{3}{2} \int \limits_\Omega h(x) \exp(-ct) \left[ \frac{\partial^3 a}{\partial t^3} - c \frac{\partial a}{\partial t} + \frac{\partial^2 a}{\partial t^2} - c \frac{\partial^2 a}{\partial t^2} \right] \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
- \int \limits_\Omega h(x) \exp(-ct) a \left| \frac{\partial^3 v^*}{\partial t^3} \right|^2 dx dt \\
+ \frac{3}{2} \int \limits_\Omega h(x) \exp(-ct) \left| \frac{\partial^3 a}{\partial t^3} \right| \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\
+ \int \limits_\Omega h(x) \exp(-ct) \left| \frac{\partial^3 a}{\partial t^3} \right| \left| \frac{\partial^4 u}{\partial t^3 \partial x} \right|^2 dx dt \\
+ \frac{1}{2} \int \limits_\Omega h(x) \exp(-ct) \frac{\partial a}{\partial t} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \right\}. \tag{4.17}
\end{align}

Integrating the first term on the right-hand side by parts in (4.15), we obtain
\begin{align}
- \epsilon \text{Re} \int \limits_\Omega A(t) u \frac{\partial^3 v^*}{\partial t^3} dx dt \\
= \frac{3}{2} \int \limits_\Omega h(x) \exp(-ct) \left[ \frac{\partial a}{\partial t} - ca \right] \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
- \int \limits_\Omega h(x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 3c \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right\} \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\
- \int \limits_0^1 \frac{1}{4} h(x) \exp(-ct) a \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx|_{t=T} \\
+ \int \limits_0^1 \frac{1}{2} h(x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a \right\} \left| \frac{\partial u}{\partial x} \right|^2 dx|_{t=T} \\
- \int \limits_0^1 h(x) \exp(-ct) \left\{ \frac{\partial a}{\partial t} - ca \right\} \left| \frac{\partial^3 u}{\partial t^3 \partial x} \right|^2 dx|_{t=T}. \tag{4.18}
\end{align}

This last equality gives
\begin{align}
- \epsilon \text{Re} \int \limits_\Omega A(t) u \frac{\partial^3 v^*}{\partial t^3} dx dt \\
\leq - \int \limits_0^1 h(x) \exp(-ct) \left| \frac{\partial a}{\partial t} + a - ca \right| \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx|_{t=T} \tag{4.19}
\end{align}

By using the conditions (3.2), inequalities (4.17) and (4.19), we obtain
\begin{align}
\text{Re} \int \limits_\Omega \exp(ct) v^* N v dx dt \leq 0 \quad \text{as} \; \epsilon \rightarrow 0. \tag{4.20}
\end{align}
This implies $\Re \int_0^T \int_0^l \exp(-ct)(1 - l)|v|^2 \, dx \, dt$

\[+ \int_0^T \int_l^1 \exp(-ct)(1 - x)|v|^2 \, dx \, dt + \int_0^T \int_0^1 \exp(-ct) \left| J_x \right|^2 \, dx \, dt \]

(4.21)

\[+ \int_0^T \int_0^l \frac{1 - l}{2l} \exp(-ct) \left| J_x \right|^2 \, dx \, dt \leq 0. \]

Then $v = 0$.

Finally from (4.4), we conclude $w = 0$. □

**Theorem 4.2.** The range $R(\bar{L})$ of $\bar{L}$ coincides with $F$.

**Proof.** Since $F$ is Hilbert space, then $R(\bar{L}) = F$ if and only if the relation

\[\int_\Omega \Theta(x) f \bar{g} \, dx \, dt = 0\] (4.22)

holds.

Arbitrary $u \in D_0(L)$ and $\bar{F} = (f, 0, 0, 0) \in F$ implies $f = 0$. Taking in (4.22), $u \in D_0(L)$, and using Lemma 4.1, we obtain

\[w = \begin{cases} (1 - l)g, & 0 < x < l, \\ (1 - x)g, & l < x < 1, \end{cases} \] (4.23)

then $g = 0$. □

**References**


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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