We establish conditions that guarantee Fredholm solvability in the Banach space $L_p$ of nonlocal boundary value problems for elliptic abstract differential equations of the second order in an interval. Moreover, in the space $L_2$ we prove in addition the coercive solvability, and the completeness of root functions (eigenfunctions and associated functions). The obtained results are then applied to the study of a nonlocal boundary value problem for Laplace equations in a cylindrical domain.

1. Introduction

Fredholm property of boundary value problems is investigated in [1, 2, 3] for elliptic partial differential equations, and in [4, 13, 14, 15] for abstract differential equations.

In this paper, we establish conditions guaranteeing that nonlocal boundary value problems for elliptic partial differential equations of the second order in an interval are Fredholm solvable in the Banach spaces $L_p$. For the solution of the considered problem we prove the noncoercive estimates. But in the space $L_2$ we prove the coercive estimate for both the variable space and spectral parameter, in contrast to [15, 16] where we have a defect coerciveness for the spectral parameter. A coercive estimate, in the case when the problem is regular elliptic, was proved in [2, 3]. The considered problem is not regular, since the boundary value conditions are nonlocal and they do not belong to the same class of boundary value conditions treated in [15, 16]. Moreover, we prove the completeness of root functions. The completeness of root functions of regular boundary value problems was proved in [1, 5, 7, 10, 13]. The obtained results are then applied to the study of a nonlocal boundary value problem for Laplace equation in a cylindrical domain.

2. Necessary notations and definitions

Let $H$ be a Hilbert space, $A$ a linear closed operator in $H$ and $D(A)$ its domain. We denote by $B(H)$ the space of bounded operators acting in $H$, with the usual operator
norm, and by \( L_p((0,1),H) \) the Banach space of strongly measurable functions \( x \mapsto u(x) : (0,1) \to H \), whose \( p \)th power is summable, with the norm
\[
\|u\|_{0,p} = \|u\|_{L_p((0,1),H)} = \int_0^1 \|u(x)\|_H^p \, dx < \infty, \quad p \in (1, \infty).
\] (2.1)

Now, introduce the \( L_p((0,1),H) \) vector-valued Sobolev spaces
\[
W^2_p((0,1),H(A),H) = \left\{ u : u'' \in L_p((0,1),H) \text{ and } Au \in L_p((0,1),H) \right\},
\]
\[
\|u\|_{W^2_p((0,1),H(A),H)} = \|Au\|_{L_p((0,1),H)} + \|u''\|_{L_p((0,1),H)} < \infty.
\] (2.2)

We also set
\[
H(A) = \left\{ u \in D(A) ; \|u\|_{2,H(A)}^2 = \|u\|_H^2 + \|Au\|_H^2 < \infty \right\},
\] (2.3)
that is, \( H(A) \) is the domain of \( A \) with a Hilbert graph norm.

Let \( -A \) be the generator of the semigroup \( \exp(-xA) \) analytic for \( x > 0 \), decreasing at infinity, and strongly continuous for \( x \geq 0 \). We define the interpolation space \([12, \text{page } 96]\)
\[
(H,H_{1});_{\theta,p} = \left\{ u : u \in H, \|u\|_{\theta,p}^p = \int_0^1 t^{-n(1-\theta)p-1}\|A^n\exp(-tA)u\|^p dt + \|u\|^p < \infty \right\}, \tag{2.4}
\]
where \( 0 < \theta < 1; \ n \in \mathbb{N}, 1 \leq p < \infty \) and \( \|\cdot\|_{\theta,p} \) its norm.

Let \( H \) and \( H_1 \) be Hilbert spaces such that the continuous embedding \( H_1 \subset H \) is fulfilled and \( H^* = H \). Then, \( (H,H_1);_{\theta,2} \) is a Hilbert space \([12, \text{page } 142]\). Denote \((H,H_1);_{\theta} = (H,H_1);_{\theta,2}\). It is known that \((H,H_1);_{\theta} = H(S^\theta)\), where \( S \) is a selfadjoint positive-definite operator in \( H \) \([11, \text{Chapter } 1, \text{Section } 2.1]\).

Let \( Ff = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} e^{ixx} f(x) \, dx \) be the Fourier transform.

**Definition 2.1.** The mapping \( \sigma \mapsto T(\sigma) : \mathbb{R} \to B(H) \) is said to be a Fourier multiplier of the type \((p,q)\) if for all \( f \in L_p(\mathbb{R},H) \) we have
\[
\|F^{-1}TFf\|_{L_q(\mathbb{R},H)} \leq c\|f\|_{L_p(\mathbb{R},H)} \quad \text{for } f \in L_p(\mathbb{R},H). \tag{2.5}
\]

We get the following characterization for Fourier multipliers.

**Theorem 2.2** (Mikhlin-Schwartz \([6, \text{page } 1181]\)). If the mapping \( T : \mathbb{R} \to B(H) : \sigma \mapsto T(\sigma) \) is continuously differentiable and the inequality
\[
\|T(\sigma)\| \leq C, \quad \left\| \frac{\partial T(\sigma)}{\partial \sigma} \right\| \leq \frac{C}{|\sigma|}, \tag{2.6}
\]
holds for all \( \sigma \in \mathbb{R}, \sigma \neq 0 \), then \( T(\sigma) \) is a Fourier multiplier of type \((p,p)\).
Lemma 2.3 (see [16, page 300]). Let \( A \) be a selfadjoint and positive-definite operator in \( H \). Then

1. \( \exists \omega > 0, \| A^\alpha \exp[−x(A + \lambda I)^{1/2}] \| \leq C \exp(−\omega x|\lambda|^{1/2}) \) for all \( \alpha \in \mathbb{R}, x \geq x_0 > 0, |\arg \lambda| \leq \varphi < \pi \), where \( C \) does not depend on \( x \) and \( \lambda \);

2. \( \int_0^1 \| (A + \lambda I)^\alpha \exp[−x(A + \lambda I)^{1/2}]u \|^2 dx \leq C(\| A^{\alpha-1/4}u \|^2 + |\lambda|^{-1/2} \| u \|^2) \) for all \( \alpha \geq 1/4, |\arg \lambda| \leq \varphi < \pi, u \in D(A^{\alpha-1/4}) \), where \( C \) does not depend on \( u \) and \( \lambda \);

3. \( \| A^\alpha (A + \lambda I)^{-\beta} \| \leq C(1 + |\lambda|)^{\alpha-\beta} \) for all \( 0 \leq \alpha \leq \beta \), where \( C \) does not depend on \( \lambda \).

3. Solvability of the principal problem

Consider in the Hilbert space \( H \) a boundary value problem in \([0, 1]\) for the second order elliptic differential-operator equation

\[
L(\lambda, D)u = −u''(x) + Au(x) + B_1(x)u'(x) + B_2(x)u(x) + \lambda u(x) = f(x), \quad (3.1)
\]

\[
L_ku = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j \leq n_k} T_{k1}u'(x_{k1j}) + \sum_{j \leq n_0} T_{k0}u(x_{k0j}) = f_k, \quad (3.2)
\]

\( k = 1, 2 \), where \( x, x_{kij} \in [0, 1]; m_k \in \{0, 1\}; \alpha_k, \beta_k \) are complex numbers; \( A, B_1(x), B_2(x), T_{ki} \) are, generally speaking, unbounded operators in \( H \); \( D = d/dx \).

First, consider the principal part of problem (3.1), (3.2)

\[
L_0(\lambda, D)u = −u''(x) + (A + \lambda I)u(x) = 0, \quad (3.3)
\]

\[
L_{k0}u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2. \quad (3.4)
\]

Theorem 3.1. Let the following conditions be satisfied:

1. \( A \) is a selfadjoint and positive-definite operator in \( H \);
2. \( \theta = (-1)^{m_1} \alpha_1 \beta_2 - (-1)^{m_2} \alpha_2 \beta_1 \neq 0 \).

Then, problem (3.3), (3.4) for \( f_k \in (H(A), H)_{\theta_k, p} \), where \( \theta_k = m_k/2 + 1/2p, p \in (1, \infty) \), and \( |\lambda| \) sufficiently large such that \( |\arg \lambda| \leq \varphi < \pi \) has a unique solution that belongs to the space \( W_2^2((0, 1), H(A), H) \), in addition for this solution we have the noncoercive estimate

\[
\| u'' \|_{L^p((0, 1), H)} + \| Au \|_{L^p((0, 1), H)} \leq C(\lambda) \sum_{k=1}^2 \| f_k \|_{(H(A), H)_{\theta_k, p}}, \quad (3.5)
\]

where \( C(\lambda) \) does not depend on \( u \).

Proof. We prove that any solution to (3.3), belonging to \( W_2^2((0, 1), H(A), H) \), has the form

\[
u(x) = \exp\left[-x(A + \lambda I)^{1/2}\right]g_1 + \exp\left[-(1-x)(A + \lambda I)^{1/2}\right]g_2. \quad (3.6)
\]
where \( g_k \in (H(A), H)_{1/2,p,p} \). To show this, let \( u \in W_{p}^{2}((0, 1), H(A), H) \) be a solution to (3.3). Then, from (3.3) we have

\[
\left[ D - (A + \lambda I)^{1/2} \right] \left[ D + (A + \lambda I)^{1/2} \right] u(x) = 0.
\]  (3.7)

Denote

\[
v(x) = \left[ D + (A + \lambda I)^{1/2} \right] u(x). \tag{3.8}
\]

Then, by virtue of Theorem 1.7 [13, page 168], \( v \in W_{p}^{1}((0, 1), H(A^{1/2}), H) \), and

\[
\left[ D - (A + \lambda I)^{1/2} \right] v(x) = 0.
\]  (3.9)

Hence,

\[
v(x) = \exp\left[ -(1 - x)(A + \lambda I)^{1/2} \right] v(1), \tag{3.10}
\]

where, in view of [12, page 44], \( v(1) \in (H(A^{1/2}), H)_{1/2,p,p} \). From (3.8) and (3.10), we have

\[
u(x) = \exp\left[ -x(A + \lambda I)^{1/2} \right] u(0)
\]

\[
+ \int_{0}^{x} \exp\left[ -(x - y)(A + \lambda I)^{1/2} \right] \exp\left[ -(1 - y)(A + \lambda I)^{1/2} \right] v(1) dy
\]

\[
= \exp\left[ -x(A + \lambda I)^{1/2} \right] u(0)
\]

\[
+ \frac{1}{2} (A + \lambda I)^{-1/2} \left\{ \exp\left[ -(x - 1)(A + \lambda I)^{1/2} \right] - \exp\left[ -x(A + \lambda I)^{1/2} \right] \exp\left[ -(A + \lambda I)^{1/2} \right] v(1) \right\}.
\]  (3.11)

where, by virtue of [12, page 44], \( u(0) \in (H(A), H)_{1/2,p,p} \). In view of [12, page 101], the operator \( A^{1/2} \) is an isomorphism from \( (H(A), H)_{1/2,p,p} = (H, H(A))_{1/2,p,p} \) onto \( (H, H(A))_{(p-1)/2,p} = (H, H(A^{1/2}))_{1/2,p} = (H(A^{1/2}), H)_{1/2,p,p} \). Thus, (3.11) has the desired form (3.6).

Let us now prove the converse, that is, the function \( u(x) \) of the form (3.6), where \( g_k \in (H(A), H)_{1/2,p,p} \), belongs to the space \( W_{p}^{2}((0, 1), H(A), H) \). Using the properties of the interpolation spaces [12, page 96], and from (3.6) we have

\[
\|u\|_{W_{p}^{2}((0,1),H(A),H)}
\]

\[
\leq \left( \left\| A(A + \lambda I)^{-1} \right\| + 1 \right) \left\{ \left( \int_{0}^{1} \left\| (A + \lambda I) \exp\left[ -x(A + \lambda I)^{1/2} \right] g_{1} \right\|^{p} dx \right)^{1/p} + \left( \int_{0}^{1} \left\| (A + \lambda I) \exp\left[ -(x - 1)(A + \lambda I)^{1/2} \right] g_{2} \right\|^{p} dx \right)^{1/p} \right\}
\]

\[
\leq C \left( \| g_{1} \|_{(H(A + \lambda I), H)_{1/2,p,p}} + \| g_{2} \|_{(H(A + \lambda I), H)_{1/2,p,p}} \right)
\]

\[
\leq C(\lambda) \left( \| g_{1} \|_{(H(A), H)_{1/2,p,p}} + \| g_{2} \|_{(H(A), H)_{1/2,p,p}} \right).
\]  (3.12)
A function \( u(x) \) of the form (3.6) satisfies the boundary condition (3.4) if
\[
(-1)^{m_k} \left\{ \alpha_k + \beta_k \exp \left[ - (A + \lambda I)^{1/2} \right] \right\} (A + \lambda I)^{m_k/2} v_1 \\
+ \left\{ \alpha_k \exp \left[ - (A + \lambda I)^{1/2} \right] + \beta_k \right\} (A + \lambda I)^{m_k/2} v_2 = f_k, \quad k = \overline{1, 2}.
\]
\[
(3.13)
\]
Denote
\[
v_1 = (A + \lambda I)^{m/2} g_1, \quad v_2 = (A + \lambda I)^{m/2} g_2,
\]
where \( m = \max\{m_1, m_2\} \). Then, (3.13) gives
\[
(-1)^{m_k} \left\{ \alpha_k + \beta_k \exp \left[ - (A + \lambda I)^{1/2} \right] \right\} (A + \lambda I)^{m_k/2 - m/2} v_1 \\
+ \left\{ \alpha_k \exp \left[ - (A + \lambda I)^{1/2} \right] + \beta_k \right\} (A + \lambda I)^{m_k/2 - m/2} v_2 = f_k, \quad k = \overline{1, 2}.
\]
\[
(3.15)
\]
All coefficients in system (3.15) are linear combinations of the bounded operators \( I, (A + \lambda I)^{-1}, \exp[-(A + \lambda I)^{1/2}], \) and \((A + \lambda I)^{-1} \exp[-(A + \lambda I)^{1/2}]\) which commute with one another. Therefore system (3.15) can be solved as in the scalar case. By virtue of Lemma 2.3, the determinant of the system (3.15) has the form
\[
D(\lambda) = \theta(A + \lambda I)^{m_1/2 + m_2/2} [I + R(\lambda)],
\]
\[
(3.16)
\]
where \( R(\lambda) = C_1 \exp[-(A + \lambda I)^{1/2}] + C_2 \exp[-2(A + \lambda I)^{1/2}] \), then by virtue of Lemma 2.3 \( \|R(\lambda)\| \to 0 \), for \(|\arg \lambda| \leq \varphi < \pi \) and \(|\lambda| \to \infty \). Then the second condition in our hypothesis implies that system (3.15) has a unique solution for \(|\arg \lambda| \leq \varphi < \pi \) and \(|\lambda| \) is sufficiently large, and the solution can be expressed in the form
\[
v_k = \left[ C_{1k}(A + \lambda I)^{-m_1/2} + R_{1k}(\lambda) \right] f_1 \\
+ \left[ C_{2k}(A + \lambda I)^{-m_2/2} + R_{2k}(\lambda) \right] f_2, \quad k = \overline{1, 2},
\]
\[
(3.17)
\]
where \( C_{jk} \) are complex numbers and \( \|R_{jk}(\lambda)\| \to 0, \ |\lambda| \to \infty \). Consequently,
\[
g_k = \left[ C_{1k}(A + \lambda I)^{-m_1/2} + (A + \lambda I)^{-m_2/2} R_{1k}(\lambda) \right] f_1 \\
+ \left[ C_{2k}(A + \lambda I)^{-m_2/2} + (A + \lambda I)^{-m_2/2} R_{2k}(\lambda) \right] f_2, \quad k = \overline{1, 2}.
\]
\[
(3.18)
\]
Since \( f_k \in (H(A), H)_{m_k/2 + 1/2, p, p} \) and the operator \((A + \lambda I)^{m_k/2}\) is an isomorphism from the space \((H(A), H)_{1/2, p, p}\) onto the space \((H, H(A))_{1 - m_k/2 - 1/2, p, p} = (H(A), H)_{m_k/2 + 1/2, p, p}\). We have \( g_k \in (H(A), H)_{1/2, p, p} \). Hence from (3.18) we have the estimate
\[
\|g_k\|_{(H(A), H)_{1/2, p, p}} \leq C(\lambda) \sum_{k=1}^{n} \|f_k\|_{(H(A), H)_{\eta_k, p}}.
\]
\[
(3.19)
\]
Substituting (3.19) in (3.12), we obtain the noncoercive estimate (3.5). \( \square \)
Consider now the principal part of problem (3.1), (3.2) for a nonhomogeneous equation and with a parameter

\[ L_0(\lambda, D)u = -u''(x) + (A + \lambda I)u(x) = f(x), \quad (3.20) \]
\[ L_{k0}u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2. \quad (3.21) \]

**Theorem 3.2.** Let the conditions of Theorem 3.1 be satisfied. Then, the operator

\[ u \mapsto (L_0(\lambda, D)u, L_{10}u, L_{20}u) \]

for \( |\arg \lambda| \leq \varphi < \pi \) and \( |\lambda| \) sufficiently large is an isomorphism from the space \( W^2_p((0, 1), H(A), H) \) onto the space \( \mathbb{H}^1_{\vartheta_1, p} \oplus (H(A), H)^{\oplus 2} \), where \( \theta_k = m_k/2 + 1/2 \), \( k = 1, 2, p \in (1, \infty) \), and for this solution we have the noncoercive estimate

\[ \|u''\|_{L^p((0, 1), H)} + \|Au\|_{L^p((0, 1), H)} \leq C(\lambda) \left( \|f\|_{L^p((0, 1), H)} + 2 \sum_{k=1}^2 \|f_k\|_{(H(A), H)^{\theta_k, p}} \right), \quad (3.22) \]

where \( C(\lambda) \) does not depend on \( u \).

**Proof.** By Theorem 3.1, we get the unicity. Now, let us define \( \tilde{f}(x) = f(x) \) if \( x \in [0, 1] \) and \( \tilde{f}(x) = 0 \) if \( x \notin [0, 1] \). We now show that a solution to problem (3.20), (3.21) belonging to \( W^2_p((0, 1), H(A), H) \) can be represented as a sum of the form

\[ u(x) = u_1(x) + u_2(x), \]

where \( u_1(x) \) is the restriction on \([0, 1]\) of the solution \( \tilde{u}_1(x) \) to the equation

\[ L_0(\lambda, D)\tilde{u}_1 = \tilde{f}(x), \quad x \in \mathbb{R}, \quad (3.23) \]

and \( u_2(x) \) is a solution to the problem

\[ L_0(\lambda, D)u_2 = 0, \quad L_{k0}u_2 = f_k - L_{k0}u_1, \quad k = 1, 2. \quad (3.24) \]

The solution to (3.23) is given by the formula

\[ \tilde{u}_1(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\mu x} L_0(\lambda, i\mu)^{-1} F \tilde{f}(\mu) d\mu, \quad (3.25) \]

where \( F \tilde{f} \) is the Fourier transform of the function \( \tilde{f}(x) \), \( L_0(\lambda, S) \) is a characteristic operator pencil of (3.23), that is,

\[ L_0(\lambda, S) = -S^2 I + A + \lambda I. \quad (3.26) \]

From (3.25), it follows that

\[ \|\tilde{u}_1\|_{W^2_p(\mathbb{R}, H(A), H)} = \|\tilde{u}_1\|_{L^p(\mathbb{R}, H(A))} + \|\tilde{u}_1''\|_{L^p(\mathbb{R}, H)} \]
\[ \leq \|F^{-1}L_0(\lambda, i\mu)^{-1}F \tilde{f}(\mu)\|_{L^p(\mathbb{R}, H(A))} + \|F^{-1}(i\mu)^2L_0(\lambda, i\mu)^{-1}F \tilde{f}(\mu)\|_{L^p(\mathbb{R}, H)} \]
\[ \leq \|F^{-1}AL_0(\lambda, i\mu)^{-1}F \tilde{f}(\mu)\|_{L^p(\mathbb{R}, H)} + \|F^{-1}(i\mu)^2L_0(\lambda, i\mu)^{-1}F \tilde{f}(\mu)\|_{L^p(\mathbb{R}, H)}, \quad (3.27) \]
Let us show that the functions
\[ T_{k+1}(\lambda, \mu) = (i \mu)^{2k} A^{1-k} L_0(\lambda, i \mu)^{-1}, \quad k = 0, 1, \] (3.28)
are Fourier multipliers in the space \( L_p(\mathbb{R}, H) \). By virtue of Lemma 2.3, for \(|\arg \lambda| \leq \varphi < \pi, |\lambda| \) sufficiently large and \( \mu \in \mathbb{R} \) we have
\[ \| L_0(\lambda, i \mu)^{-1} \| = \| (A + \lambda I + \mu^2 I)^{-1} \| \leq C (1 + |\lambda + \mu^2|)^{-1} \leq C |\mu|^{-2}, \] (3.29)
\[ \| A L_0(\lambda, i \mu)^{-1} \| = \| A (A + \lambda I + \mu^2 I)^{-1} \| \leq C. \] (3.30)
From (3.29) and (3.30), we get
\[ \| T_1(\lambda, \mu) \|_{B(H)} \leq C \| A L_0(\lambda, i \mu)^{-1} \|_{B(H)} \leq C, \] (3.31)
\[ \| T_2(\lambda, \mu) \|_{B(H)} \leq C |\mu|^2 \| L_0(\lambda, i \mu)^{-1} \|_{B(H)} \leq C. \] (3.32)
Since
\[ \frac{\partial}{\partial \mu} T_{k+1}(\lambda, \mu) = 2k i^{2k} \mu^{2k-1} A^{1-k} L_0(\lambda, i \mu)^{-1} \]
\[ - i^{2k+1} \mu^{2k} A^{1-k} L_0(\lambda, i \mu)^{-1} \frac{\partial}{\partial \mu} L_0(\lambda, i \mu) L_0(\lambda, i \mu)^{-1}, \] (3.33)
then,
\[ \left\| \frac{\partial}{\partial \mu} T_{k+1}(\lambda, \mu) \right\| \leq |\mu|^{-1}. \] (3.34)
Applying the Michlin-Schwartz Theorem 2.2, it follows from (3.31), (3.32), and (3.34) that the functions \( \mu \to T_{k+1}(\lambda, \mu) \) are Fourier multipliers in the space \( L_p(\mathbb{R}, H) \). Then, using (3.27), we obtain
\[ \| \tilde{u}_1 \|_{W^2_p(\mathbb{R}, H(A), H)} \leq C \| \tilde{f}_1 \|_{L_p(\mathbb{R}, H)}. \] (3.35)
So, \( u_1 \in W^2_p((0, 1), H(A), H) \). By virtue of [12, page 44] and inequality (3.35), we have \( u_1^{m_k}(0) \in (H(A), H)_{m/2+1/2p,p} \). Hence, \( L_{k0} u_1 \in (H(A), H)_{\theta_k,p} \). Then by virtue of Theorem 3.1, problem (3.20), (3.21) has a unique solution \( u_2(x) \) that belongs to \( W^2_p((0, 1), H(A), H) \). And, again, by Theorem 3.1 and estimate (3.35), we obtain the inequality (3.22).

4. Fredholm solvability of general problem

Consider problem (3.1), (3.2). Now we can find conditions for the Fredholm solvability of problem (3.1), (3.2). It is convenient to formulate the theorem in terms of the Fredholmness of some unbounded operator which acts from one Banach space into another.
Let us set the operator \( L \) from \( W^2_p((0,1), H(A), H) \) into \( L_p((0,1), H) \oplus (H(A), H)_{\theta_1, p} \oplus (H(A), H)_{\theta_2, p} \), by the equalities

\[
D(L) = \left\{ \frac{u'}{u} \in W^2_p((0,1), H(A), H), \quad L(D)u \in L_p((0,1), H), \quad L_k u \in (H(A), H)_{\theta_1, p}, \quad k = 1, 2 \right\},
\]

\[
\mathbb{L}u = (L(D)u, L_1 u, L_2 u),
\]

where

\[
L(D)u = -u''(x) + Au(x) + B_1(x)u'(x) + B_2(x)u(x),
\]

\[L_1 \text{ and } L_2 \text{ have been defined by equalities (3.2).}
\]

**Theorem 4.1.** Suppose that in addition to conditions (1), (2) stated in Theorem 3.1, the following conditions are also satisfied:

1. the embedding \( H(A) \subset H \) is compact;
2. linear operators \( B_k(x) \) from \( H(A^{k/2}) \) into \( H \) act compactly for almost all \( x \in [0, 1] \); for any \( \varepsilon > 0 \) and for almost all \( x \in [0, 1] \),

\[
\| B_k(x)u \| \leq \varepsilon \| A^{k/2}u \| + c(\varepsilon)\|u\|, \quad u \in D(A^{k/2});
\]

\[
(4.3)
\]

3. the linear operators \( T_{ki} \) from \( (H(A), H)_{(\theta_{k+1})/2, p} \) into \( (H(A), H)_{\theta_k, p} \) are compact, where \( \theta_k = m_k/2 + 1/2p, \quad p \in (1, \infty) \).

Then,

1. for any function \( u(x) \in W^2_p((0,1), H(A), H) \), we have the noncoercive estimate

\[
\| u'' \|_{L_p((0,1), H)} + \| Au \|_{L_p((0,1), H)}
\]

\[
\leq C(\lambda) \left( \| L(D)u \|_{L_p((0,1), H)} + \sum_{k=1}^2 \| L_k u \|_{(H(A), H)_{\theta_k, p}} + \| u \|_{L_p((0,1), H)} \right),
\]

\[
(4.4)
\]

where \( C(\lambda) \) does not depend on \( u \); (4.4)

2. the operator \( \mathbb{L} : u \mapsto (L(D)u, L_1 u, L_2 u) \), from \( W^2_p((0,1), H(A), H) \) into \( L_p((0,1), H) \oplus (H(A), H)_{\theta_1, p} \oplus (H(A), H)_{\theta_2, p} \), is Fredholm.

**Proof.** (1) Let \( u(x) \) be a solution to problem \( L(D)u = f, \quad L_k u = f_k, \quad k = 1, 2 \), belonging to \( W^2_p((0,1), H(A), H) \). Then, \( u(x) \) is a solution to the problem

\[
L_0(\lambda, D)u = f(x) + \lambda u(x) - B_1(x)u'(x) - B_2(x)u(x),
\]

\[
L_{k0} u = f_k - \sum_{j \leq n_k} T_{k1} u'(x_{k1j}) - \sum_{j \leq n_k} T_{k0} u(x_{k0j}); \quad k = 1, 2.
\]

\[
(4.5)
\]
where \( L_0(\lambda, D) \) and \( L_{k0} \) have been defined by (3.20) and (3.21). By virtue of Theorem 3.2, for some \( \lambda_0 \), we have the estimate
\[
\|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)} 
\leq C(\lambda_0) \left( \|f(\cdot) + \lambda_0 u(\cdot) - B_1(\cdot)u'(\cdot) - B_2(\cdot)u(\cdot)\|_{L_p((0,1),H)} 
+ \sum_{k=1}^{2} \|f_k - \sum_{j \leq n_k} T_{k1}u'(x_{k1j}) - \sum_{j \leq n_{k0}} T_{k0}u(x_{k0j})\|_{(H(A),H)_{\theta_k,p}} \right).
\] (4.6)

By virtue of Theorem 5.1.7 [13, page 168], the operator \( u \mapsto u^{(i)}(x) \) from the space \( W^{2,p}_{p}((0,1),H(A),H) \) into the space \( L^{p}_{p}((0,1),H(A)^{(1-i/2)}) \) is bounded. Then, by condition (2) of Theorem 4.1 and Lemma 5.1.2 [13, page 162], the operator \( u \mapsto -\lambda_0 u - B_1(x)u'(x) - B_2(x)u^2(x) \) (4.7) from \( W^{2,p}_{p}((0,1),H(A),H) \) into the space \( L^{p}_{p}((0,1),H) \) is compact. Consequently, by Lemma 2.2.7 [13, page 53], for any \( \varepsilon > 0 \) we have
\[
\|f(\cdot) + \lambda_0 u(\cdot) - B_1(\cdot)u'(\cdot) - B_2(\cdot)u(\cdot)\|_{L_p((0,1),H)} 
\leq C(\lambda_0) \left( \|f\|_{L_p((0,1),H)} + \varepsilon \left( \|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)} \right) 
+ C(\varepsilon) \|u\|_{L_p((0,1),H)} \right). 
\] (4.8)

By virtue of [12, page 44], we have the operator \( u \mapsto u^{(i)}(x_0) \) from \( W^{2,p}_{p}((0,1),H(A),H) \) into \( (H(A),H)_{(p+1)/2,p} \) is bounded. Then, by virtue of condition (3), the operator \( u \mapsto \sum_{j \leq n_k} T_{k1}u'(x_{k1j}) + \sum_{j \leq n_{k0}} T_{k0}u(x_{k0j}) \) from \( W^{2,p}_{p}((0,1),H(A),H) \) into \( (H(A),H)_{\theta_k,p} \) is compact. Consequently, by [13], for any \( \varepsilon > 0 \) we have
\[
2 \sum_{k=1}^{2} \|f_k - \sum_{j \leq n_k} T_{k1}u'(x_{k1j}) - \sum_{j \leq n_{k0}} T_{k0}u(x_{k0j})\|_{(H(A),H)_{\theta_k,p}} 
\leq C(\lambda_0) \left( 2 \sum_{k=1}^{2} f_k(\cdot)_{(H(A),H)_{\theta_k,p}} + C(\varepsilon) \|u\|_{L_p((0,1),H)} 
+ \varepsilon \left( \|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)} \right) \right). 
\] (4.9)

Substituting (4.8) and (4.9) into (4.6) we have (4.4).

(2) The operator \( L \) can be rewritten in the form \( L = L_{0\lambda} + L_{1\lambda} \), where
\[
L_{0\lambda}u = (L_0(\lambda, D)u, L_{10}u, L_{20}u), 
\] (4.10)
Conditions (2) and (3) imply that the solution $u$ from $C$ where

\[ 5.1. \quad \text{Theorem} \quad (H(A), H)_{\theta} \]

We seek a solution in the form

\[ \sum_{j \in n_{k1}} T_{kj} u(x_{kj}) + \sum_{j \in n_{k0}} T_{k0} u(x_{k0}), \quad k = 1, 2. \]  

(4.12)

Consider a particular case of problem (3.20), (3.21) in $(H(A), H)_{\theta}$ where $\theta$ from $W^2((0, 1), H(A), H)$ into $L_p((0, 1), H) \oplus (H(A), H)_{\theta_1, p}$ has an inverse for $\lambda$ is sufficiently large. It follows from the proof of part (1) that the operator $L_{\lambda}$ from $W^2((0, 1), H(A), H)$ into $L_p((0, 1), H) \oplus (H(A), H)_{\theta_1, p}$ is compact. By applying the perturbation theorem of Fredholm operators [9, page 238] to the operator $L$, we end the proof. □

5. Coercive solvability in $L_2((0, 1), H)$

Consider a particular case of problem (3.20), (3.21) in $L_2((0, 1), H)$

\[ L(\lambda, D)u = -u''(x) + Au(x) + \lambda u(x) = f(x), \]

(5.1)

\[ L_k u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2. \]  

(5.2)

**Theorem 5.1.** Let the following conditions be satisfied:

1. $A$ is a selfadjoint and positive-definite operator in a Hilbert space $H$;
2. $(1)^{m_1} \alpha_1 \beta_2 - (1)^{m_2} \alpha_2 \beta_1 \neq 0$.

Then, problem (5.1), (5.2) for $f \in L_2((0, 1), H)$, $f_k \in D(A^{3/4 + m_k/2})$ for $|\arg \lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large has a unique solution that belongs to the space $W^2((0, 1), H(A), H)$ and for this solution we have the coercive estimate

\[ |\lambda| \|u\|_{L_2((0,1), H)} + \|u''\|_{L_2((0,1), H)} + \|Au\|_{L_2((0,1), H)} \leq C \left( \|f\|_{L_2((0,1), H)} + \sum_{k=1}^{2} \left( \|A^{-m_k/2 + 3/4} f_k\| + |\lambda|^{-m_k/2 + 3/4} \|f_k\| \right) \right), \]

(5.3)

where $C$ does not depend on $u$, $f$, $f_k$, and $\lambda$.

**Proof.** We seek a solution in the form $u = u_1 + u_2$, where $u_1$ is the restriction on $[0, 1]$ of the solution $\hat{u}_1(x)$ to (3.23) and $u_2$ is a solution of problem (3.24). From Theorem 3.1, we have

\[ u_2(x) = \left[ C_{11}(A + \lambda I)^{-m_1/2} + (A + \lambda I)^{-m_2/2} R_{11}(\lambda) \right] \exp \left[ -x(A + \lambda I)^{1/2} \right] \]

\[ + \left[ C_{12}(A + \lambda I)^{-m_1/2} + (A + \lambda I)^{-m_2/2} R_{12}(\lambda) \right] \exp \left[ -(1-x)(A + \lambda I)^{1/2} \right] \]

\[ + \left[ C_{21}(A + \lambda I)^{-m_2/2} + (A + \lambda I)^{-m_2/2} R_{21}(\lambda) \right] \exp \left[ -x(A + \lambda I)^{1/2} \right] \]

\[ + \left[ C_{22}(A + \lambda I)^{-m_2/2} + (A + \lambda I)^{-m_2/2} R_{22}(\lambda) \right] \exp \left[ -(1-x)(A + \lambda I)^{1/2} \right] \]

(5.4)
Then, for $|\arg \lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large, we have

$$
|\lambda| \|u_2\|_{L^2((0,1),H)} + \|u''_2\|_{L^2((0,1),H)} + \|Au_2\|_{L^2((0,1),H)} \leq C \left( \|f\|_{L^2((0,1),H)} + \sum_{k=1}^{2} \|A^{-m_k/2+3/4}f_k\| + |\lambda|^{-m_k/2+3/4} \|f_k\| \right). \tag{5.5}
$$

From (3.25) and the Plancherel equality, we have

$$
|\lambda| \|u_1\|_{L^2((0,1),H)} + \|u''_1\|_{L^2((0,1),H)} + \|Au_1\|_{L^2((0,1),H)} \leq |\lambda| \|\hat{u}_1\|_{L^2(\mathbb{R},H)} + \|\hat{u''}_1\|_{L^2(\mathbb{R},H)} + \|A\hat{u}_1\|_{L^2(\mathbb{R},H)}
= |\lambda| \|L_0(\lambda, i\mu)^{-1}(F\hat{f})(\mu)\|_{L^2(\mathbb{R},H)} + \|(i\mu)^2L_0(\lambda, i\mu)^{-1}(F\hat{f})(\mu)\|_{L^2(\mathbb{R},H)} + \|AL_0(\lambda, i\mu)^{-1}(F\hat{f})(\mu)\|_{L^2(\mathbb{R},H)} \tag{5.6}
$$

From condition (1) of Theorem 5.1, for $|\arg \lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large, we have

$$
|\lambda| \|L_0(\lambda, i\mu)^{-1}\| = |\lambda| \|(A + \lambda I + \mu^2 I)^{-1}\| \leq c|\lambda|(1 + |\lambda + \mu^2|)^{-1} \leq C, \tag{5.7}
$$

$$
|\mu|^2 \|L_0(\lambda, i\mu)^{-1}\| = |\mu|^2 \|(A + \lambda I + \mu^2 I)^{-1}\| \leq c|\mu|^2(1 + |\lambda + \mu^2|)^{-1} \leq C.
$$

Then, from (5.6), it follows that

$$
|\lambda| \|u_1\|_{L^2((0,1),H)} + \|u''_1\|_{L^2((0,1),H)} + \|Au_1\|_{L^2((0,1),H)} \leq C \|f\|_{L^2((0,1),H)}. \tag{5.8}
$$

From [12, page 44], we have

$$
\|A^{-m_k/2+3/4}u_1^{(m_k)}(0)\| \leq C \|u_1\|_{W^{2,\infty}_{1,\infty}(0,1),H(A),H} \leq C \|f\|_{L^2((0,1),H)}, \tag{5.9}
$$

we also use the inequality [11, Chapter 1, Section 3.2]

$$
\|u^{(j)}(0)\|_H \leq C \left( h^{1-\chi} \|u\|_{W^{2,\infty}_{1,\infty}(0,1),H(A),H} + h^{-\chi} \|u\|_{L^2((0,1),H)} \right), \tag{5.10}
$$

where $0 \leq j \leq 1$, $0 < h < h_0$, $\chi = j + (1/2)/2$. Then

$$
|\lambda|^{-m_k/2+3/4} \|u^{(m_k)}(0)\| \leq C|\lambda|^{-m_k/2+3/4} \left( h^{1-(m_k/2+1/4)} \|u\|_{W^{2,\infty}_{1,\infty}(0,1),H(A),H} + h^{-m_k/2-1/4} \|u\|_{L^2((0,1),H)} \right), \tag{5.11}
$$

by taking $h = |\lambda|^{-1}$, and so from (5.8) and (5.9), we have

$$
|\lambda|^{-m_k/2+3/4} \|u^{(m_k)}(0)\| \leq C \left( \|u\|_{W^{2,\infty}_{1,\infty}(0,1),H(A),H} + |\lambda| \|u\|_{L^2((0,1),H)} \right) \leq \|f\|_{L^2((0,1),H)}, \tag{5.12}
$$
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From (5.6), (5.9), and (5.12) we get
\[ |\lambda| u_2 \|_{L^2((0,1),H)} + \| u'' \|_{L^2((0,1),H)} + \| Au \|_{L^2((0,1),H)} \leq C \left( \sum_{k=1}^{2} \left( \| A^{-m_k/2+3/4} f_k \| + |\lambda|^{-m_k/2+3/4} \| f_k \| \right) \right). \] (5.13)

Hence, (5.3) follows from (5.8) and (5.13).

Consider in \( L^2((0,1),H) \), the following problem:
\[ L(\lambda, D)u = \lambda u(x) - u''(x) + Au(x) + B(x)u = f(x), \] (5.14)
\[ L_k u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2. \] (5.15)

**Theorem 5.2.** Let the following conditions be satisfied:

1. \( A \) is a selfadjoint and positive-definite operator in a Hilbert space \( H \);
2. the embedding \( H(A) \subset H \) is compact;
3. \((-1)^{m_1} \alpha_1 \beta_2 - (-1)^{m_2} \alpha_2 \beta_1 \neq 0;\)
4. \( \| Bu \|_{L^2((0,1),H)} \leq \varepsilon \| Au \|_{L^2((0,1),H)} + C(\varepsilon) \| u \|_{L^2((0,1),H)}. \)

Then, problem (5.14), (5.15) for \( f \in L^2((0,1),H) \), \( f_k \in D(A^{m_k/2+3/4}) \) for \( \arg \lambda \leq \phi < \pi \) and \( |\lambda| \) is sufficiently large has a unique solution that belongs to the space \( W^2_p((0,1),H(A),H) \) and for this solution we have the coercive estimate
\[ |\lambda| \| u \|_{L^2((0,1),H)} + \| u'' \|_{L^2((0,1),H)} + \| Au \|_{L^2((0,1),H)} \leq C \left( \| f \|_{L^2((0,1),H)} + \sum_{k=1}^{2} \left( \| A^{-m_k/2+3/4} f_k \| + |\lambda|^{-m_k/2+3/4} \| f_k \| \right) \right), \] (5.16)
where \( C \) does not depend on \( u, f, f_k, \) and \( \lambda \).

**Proof.** Let \( u \in W^2_p((0,1),H(A),H) \) be a solution to problem (5.14), (5.15). Then, by virtue of Theorem 5.1, we have
\[ |\lambda| \| u \|_{L^2((0,1),H)} + \| u'' \|_{L^2((0,1),H)} + \| Au \|_{L^2((0,1),H)} \leq C \left( \| f - Bu \|_{L^2((0,1),H)} + \sum_{k=1}^{2} \left( \| A^{-m_k/2+3/4} f_k \| + |\lambda|^{-m_k/2+3/4} \| f_k \| \right) \right), \] (5.17)
using condition (4) of Theorem 5.2, we have
\[ (|\lambda| - C \cdot C(\varepsilon)) \| u \|_{L^2((0,1),H)} + \| u'' \|_{L^2((0,1),H)} + (1 - C \cdot \varepsilon) \| Au \|_{L^2((0,1),H)} \leq C \left( \| f \|_{L^2((0,1),H)} + \sum_{k=1}^{2} \left( \| A^{-m_k/2+3/4} f_k \| + |\lambda|^{-m_k/2+3/4} \| f_k \| \right) \right), \] (5.18)
by choosing \( \varepsilon \) such that \( C \cdot \varepsilon < 1 \), (5.16) is easily obtained from (5.18).
6. Completeness of root functions

We define the operator $\mathcal{L}$ by

$$\mathcal{L}u = -u''(x) + Au(x), \quad \mathcal{D}(\mathcal{L}) = W^2_p((0, 1); H(A), H, L_ku = 0, k = 1, 2), \quad (6.1)$$

**Lemma 6.1.** Suppose that $S_j(I, H(A), H) \sim C_j^{-q}$, then

$$S_j(I, W^2_p((0, 1), H(A), H), L_2((0, 1), H)) \sim C_j^{-1/(1/2 + 1/q)}. \quad (6.2)$$

**Proof.** Consider the operator $S_1$ defined in $L_2(0, 1)$ such that $S_1 = S^*_1 \geq \gamma^2 I$, $D(S_1) = H(S_1) = W^2_p(0, 1)$. From [11, Chapter 1, Section 2.1], we know that if $H_1 \subset H$, $\overline{H}_1 = H$, then there exists $S = S^*$ such that $D(S_1) = H_1$. And let the operator $S_2$ in $H$ be defined by $S_2 = S^*_2 \geq \gamma^2 I$, $D(S_2) = H(A)$. If we define the operator $S$ on $L_2(0, 1) \otimes H = L_2((0, 1), H)$ by

$$S = S_1 \otimes I_2 + I_1 \otimes S_2, \quad (6.3)$$

where $I_1$ (respectively, $I_2$) is the identity operator in $L_2(0, 1)$ (respectively, in $H$), we have

$$S_j(S_1^{-1}; L_2(0, 1), L_2(0, 1)) \sim S_j(I; H(S_1), L_2(0, 1)) \sim C_j^{-2},$$

$$S_j(S_2^{-1}; H, H) \sim S_j(I; H(A), H) \sim C_j^{-q}, \quad (6.4)$$

and so, from [8], we obtain

$$S_j(S^{-1}) \sim Cn^{-1/(1/2 + 1/q)}. \quad (6.5)$$

This ends the proof. \hfill \Box

**Theorem 6.2.** Let conditions (1) and (3) of Theorem 5.2 hold along with $A^{-1} \in \sigma_q(H), q > 0$. Then, the system of root functions of operator $\mathcal{L}$ is complete in $L_2((0, 1), H)$.\hfill \Box

**Proof.** From Theorem 5.2, we have $\|R(\lambda, \mathcal{L})\| \leq C|\lambda|^{-1}$ for $|\arg \lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large. Using Lemma 6.1, we have $R(\lambda, \mathcal{L}) \in \sigma_p(L_2((0, 1), H))$, for $p > 1/2 + 1/q$, so, for the operator $\mathcal{L}$, all conditions of Theorem 2.3 [13, page 50] have been checked. This completes the proof of the theorem. \hfill \Box

**Theorem 6.3.** Suppose that the conditions of Theorem 6.2 are satisfied, as well as the condition $D(B(x)) \supset D(A)$, and for all $\varepsilon > 0$

$$\|B(x)u\| \leq \varepsilon \|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A), \quad (6.6)$$

then the system of root functions of operator $\mathcal{L} + B$ is complete in $L_2((0, 1), H)$.\hfill \Box

**Proof.** We consider in the space $L_2((0, 1), H)$ the operator $B$ defined by

$$(Bu)(x) = B(x)u(x), \quad D(B) = L_2((0, 1), H(A)). \quad (6.7)$$
It is clear that
\[ \| Bu \|_{L_2((0,1), H)} \leq \| Au \|_{L_2((0,1), H)} + C(\varepsilon) \| u \|_{L_2((0,1), H)}, \] (6.8)
since by Theorem 5.2, we have
\[ \| Au \|_{L_2((0,1), H)} \leq C \| f \|_{L_2((0,1), H)} = \| (\mathcal{L} - \lambda I)u \|_{L_2((0,1), H)}, \] (6.9)
hence,
\[ \| Bu \|_{L_2((0,1), H)} \leq \varepsilon \| (\mathcal{L} - \lambda I)u \|_{L_2((0,1), H)} + C(\varepsilon) \| u \|_{L_2((0,1), H)}, \] (6.10)
and so, for \(|\lambda|\) sufficiently large, and \(|\arg \lambda| < \varphi < \pi\), then \(R(\lambda, \mathcal{L} + B) \in \sigma_p(L_2(0,1), H)\), and from Theorem 5.2, \(\| R(\lambda, \mathcal{L} + B_2) \| \leq C|\lambda|^{-1}\) for \(|\arg \lambda| < \varphi < \pi\) and \(|\lambda|\) is sufficiently large. The system of root functions is complete in \(L_2((0,1), H)\).

7. Application

We consider in the cylindrical domain \(\Omega = [0, 1] \times G\), where \(G \subset \mathbb{R}^r\) is a bounded domain, nonlocal boundary value problems for the Laplace equation with a parameter
\[ L(\lambda)u = \lambda u(x, y) - \Delta u(x, y) + b(x, y)u(x, y) = f(x, y), \quad (x, y) \in \Omega, \] (7.1)
\[ L_ku = \alpha_k u^{(m_k)}(0, y) + \beta_k u^{(m_k)}(1, y) = f_k(y), \quad y \in G, \quad k = 1, 2, \] (7.2)
\[ Pu = u(x, y') = 0, \quad (x, y') \in [0, 1] \times \Gamma, \] (7.3)
where \(\alpha_k, \beta_k\) are complex numbers, \(y = (y_1, \ldots, y_r)\), and \(\Gamma = \partial G\) is the boundary of \(G\).

A number \(\lambda_0\) is called an eigenvalue of problem (7.1), (7.2), and (7.3) if the problem
\[ L(\lambda_0)u = 0, \quad L_1u = 0, \quad L_2u = 0, \quad Pu = 0 \] (7.4)
has a nontrivial solution that belongs to \(W^2_2(\Omega)\). The nontrivial solution \(u_0(x, y)\) of problem (7.4) that belongs to \(W^2_2(\Omega)\) is called eigenfunction of problem (7.1), (7.2), and (7.3) and corresponds to the eigenvalue \(\lambda_0\). A solution \(u_k(x)\) to the problem
\[ L(\lambda_0)u_k + u_{k-1} = 0, \quad L_1u_k = 0, \quad L_2u_k = 0, \quad Pu_k = 0 \] (7.5)
belongs to \(W^2_2(\Omega)\), and is called an associated function of the \(k\)th rank to the eigenfunction \(u_0(x)\) of problem (7.1), (7.2), and (7.3).

Eigenfunctions and associated functions of problem (7.1), (7.2), and (7.3) are gathered under the general name, root functions of problem (7.1), (7.2), and (7.3).

**Theorem 7.1.** Let \(b(x, y) \in W^{0,1}_\infty(\Omega)\), \((-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0, \Gamma \in \mathcal{C}^2\). Then,

(1) Problem (7.1), (7.2), and (7.3) for \(f \in W^{0,1}_2(\Omega), Pu = 0, f_k \in W^{-m_k/2+3/4}_2(G), Pu = 0\) for \(|\arg \lambda| \leq \varphi < \pi\) and \(|\lambda|\) is sufficiently large has a unique solution that belongs to the space \(W^2_2(\Omega)\), and for this solution we have the coercive estimate
\[ |\lambda| \| u \|_{L_2(\Omega)} + |u|_{W^2_2(\Omega)} \leq C\left( \| f \|_{L_2(\Omega)} + \sum_{k=1}^{2} \left( \| f_k \|_{W^{-m_k/2+3/4}_2} + |\lambda|^{-m_k/2+3/4} \| f_k \| \right) \right). \] (7.6)
where the constant \( C \) is independent of \( u \) and \( \lambda \).

(2) Root functions of problem (7.1), (7.2), and (7.3) are complete in the space \( L^2(\Omega) \).

Proof. Consider in the space \( H = L^2(G) \) operators \( A, B(x) \) defined by

\[
Au = -\Delta u(y) + \lambda_0 u(y), \quad D(A) = W^2_2(G; Pu = 0),
\]

\[
B(x)u = b(x, y)u(y) - \lambda_0 u(y), \quad D(B(x)) = W^1_2(G; Pu = 0, m = 0).
\]

Then, problem (7.1), (7.2), and (7.3) can be rewritten in the form

\[
\lambda u(x) - u''(x) + Au(x) + B_1(x)u'(x) + B_2(x)u = f(x),
\]

\[
\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2.
\]

We have the compact embedding \([12, \text{page 258}] W^2_2(\Omega) \subset L^2(\Omega) \). On the other hand, (cf. \([12, \text{page 350}]\))

\[
S_j(I; W^2_2(\Omega), L^2(\Omega)) \sim j^{-2/(r+1)}.
\]

By virtue of Lemma 3.1 \([13, \text{page 60}]\)

\[
S_j(I; H(A), L^2(\Omega)) = S_j(A^{-1}; L^2(\Omega), L^2(\Omega)).
\]

Since \( H(A) \subset W^2_2(\Omega) \), then, from (7.10), (7.11), and Lemma 3.3 \([13, \text{page 61}]\), it follows that

\[
A^{-1} \in \sigma_p(L^2(\Omega), L^2(\Omega)), \quad p > \frac{r+1}{2}.
\]

From (7.6) it follows that

\[
\|R(\lambda, A)\| \leq C|\lambda|^{-1}, \quad |\arg \lambda| \leq \varphi < \pi, \quad |\lambda| \text{ is sufficiently large}.
\]

So, all conditions of Theorem 6.3 have been checked. This ends the proof of the theorem. \( \square \)

References


Second order abstract elliptic differential equation


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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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