COMPLEMENTED SUBALGEBRAS OF THE BAIRE-1 FUNCTIONS DEFINED ON THE INTERVAL [0, 1]

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We prove that if the Banach algebra of bounded real Baire-1 functions (resp., small Baire class ξ), defined on $[0,1]$, is the direct sum of two subalgebras, then one of its components contains a copy of it as a complemented subalgebra.

An important problem in topology is to determine the effects on the function space of imposing some natural topological condition on the space X [5].

In this way, it is practical to characterize the complemented subalgebras of Banach algebras. In our investigation, we relate the second category property of $[0,1]$ with certain properties of the complemented subalgebras of bounded real Baire-1 functions, $\beta_1^\circ([0,1])$ (bounded functions of finite Baire index, $\mathcal{B}_1^F([0,1])$).

We begin by recalling some definitions. Let A be a Banach algebra (resp., Banach space). Two subalgebras (resp., subspaces) M and N of A are complementary if $A = M \oplus N$. A projection on A is a continuous linear operator $P : A \to A$ satisfying $P^2 = P$. If M and N are complementary subalgebras (resp., subspaces) of A, then there exists a projection P on A such that the range of P is M (N). The norm (sup-norm) of the projection P is always equal to or greater than 1. Norm 1 projections play a crucial role in the study of complemented subalgebras (resp., subspaces) of Banach algebras (resp., Banach spaces) (see, e.g., [4]). If A is a finite-dimensional Banach space, then every nontrivial subspace of A is closed and complemented in A and does not contain a copy of A. This of course is more complicated for infinite-dimensional Banach spaces.

Pelczynski has proved that every infinite-dimensional closed linear subspace of l_1 contains a complemented subspace of l_1 that is isomorphic to l_1 [4, Theorem 6, page 74]. Also it has been proved that $C(X)$, the ring of continuous functions on the compact topological space X is the direct sum of two proper subrings if and only if X is disconnected [5, Problem 1.B, page 20]. In this case, for every decomposition of $C(X)$, there is an open compact partition $\{A,B\}$ of X such that $C(X) = C(A) \oplus C(B)$. We want to establish a result similar to that of Pelczynski for the Banach algebra of bounded real Baire-1 functions defined on $[0,1]$.

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Throughout this paper, X is a compact subset of real numbers. The class of open (resp., closed) subsets of X is denoted by $\mathcal{G}$ (resp., $\mathcal{F}$). We define $\mathcal{G}_\delta$ (resp., $\mathcal{F}_\sigma$) as the class of all of countable intersections (resp., unions) of elements of $\mathcal{G}$ (resp., $\mathcal{F}$). We denote the set $(\mathcal{G}_\delta \cap \mathcal{F}_\sigma)$ by $\mathcal{H}$.

Let X be a topological space. We define the real Baire functions of class 1 as follows:

$$\begin{aligned}\beta_1(X) = \{f : X \longrightarrow \mathbb{R} : \exists\ (f_n)_{n=1}^{\infty} \subseteq C(X) \\ \text{such that } \lim f_n(x) = f(x), \text{ for each } x \in X\}.\end{aligned} \tag{1}$$

We denote the set of bounded functions in $\beta_1(X)$ by $\beta_1^\circ(X)$. The Baire-1 class, $\beta_1^\circ(X)$ has an algebraic and isometric representation as the space $C(\omega)$ of all continuous functions on a totally disconnected compact space ω. This representation was used to show that if the compact space S has an uncountable compact metrizable subset, then $\beta_1^\circ(S)$ is not linearly isomorphic to any complemented subspace of the Banach space $C(K)$ for σ-stonian space K [3]. In [1], Bade studied the linear complementation problem for the Baire classes. He proved that $\beta_\alpha([0,1])$ is not complemented as a closed subspace of $\beta_{\alpha+1}([0,1])$ for each ordinal $\alpha < \omega_1$.

In our investigation, we characterize the complemented topological subalgebras of the Baire-1 classes on $[0,1]$.

Theorem 1. *If the Banach algebra of bounded real Baire-*1 *functions, defined on* $[0,1]$, *is the direct sum of two subalgebras, then one of its components contains a copy of it as a complemented subalgebra; that is, if A and B are two subalgebras of* $\beta_1^\circ([0,1])$ *such that*

$$\beta_1^\circ([0,1]) = A \oplus B, \tag{2}$$

then there exist two subalgebras, C and D, of A (or B) such that

$$A = C \oplus D, \qquad C \cong \beta_1^\circ([0,1]). \tag{3}$$

Moreover, each complemented subalgebra of $\beta_1^\circ([0,1])$ *can be obtained by a norm*1, *positive and multiplicative projection.*

Proof. First we note that the idempotents of the ring $\beta_1^\circ([0,1])$ are χ_H's for $H \in \mathcal{H}$ [1]. It is obvious that $H \in \mathcal{H}$ if and only if $[0,1] - H \in \mathcal{H}$. Suppose that the ring $\beta_1^\circ([0,1])$ is the direct sum of two subrings A and B,

$$\beta_1^\circ([0,1]) = A \oplus B. \tag{4}$$

The constant function $\hat{1}$ belongs to $\beta_1^\circ([0,1])$, therefore there exist $e_1 \in A$ and $e_2 \in B$ such that

$$\hat{1} = e_1 + e_2. \tag{5}$$

Thus e_1 and e_2 are two disjoint idempotents; that is, $e_1e_2 = 0$, because $A \cap B = \{0\}$, and

$$e_1e_2 = e_1 - e_1^2 = e_2 - e_2^2 \in (A \cap B) = \{0\}. \tag{6}$$

Suppose that $e_1 = \chi_{H_1}$ and $e_2 = \chi_{H_2}$ for suitable H_1 and H_2 in $\mathcal{H}$. By (5), $\{H_1, H_2\}$ is a partition of $[0,1]$ by $\mathcal{H}$ sets. Therefore, we conclude that $A = \beta_1^\circ([0,1])\,\chi_{H_1} \cong \beta_1^\circ(\chi_{H_1})$ and similarly $B = \beta_1^\circ(\chi_{H_2})$. Now, suppose that $\{H_1, H_2\}$ is a partition of $[0,1]$ by $\mathcal{H}$ sets. Let f and g be in $\beta_1^\circ(H_1)$ and $\beta_1^\circ(H_2)$, respectively. We define h as follows:

$$h(x) = \begin{cases} f(x) & \text{if } x \in H_1, \\ g(x) & \text{if } x \in H_2. \end{cases} \tag{7}$$

Suppose F is a closed subset of $\mathbb{R}$. Then

$$h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F). \tag{8}$$

Hence $f^{-1}(F)$ and $g^{-1}(F)$ are $\mathcal{G}_\delta$ sets in H_1 and H_2, respectively, and so they are both $\mathcal{G}_\delta$ in $[0,1]$. Consequently $h^{-1}(F)$ is $\mathcal{G}_\delta$ in $[0,1]$ and $h \in \beta_1^\circ(X)$ [1]. It is obvious that if $h \in \beta_1^\circ(X)$, then $h|_{H_1} \in \beta_1^\circ(H_1)$ and $h|_{H_2} \in \beta_1^\circ(H_2)$. We now define φ from $\beta_1^\circ([0,1])$ onto $\beta_1^\circ(H_1) \oplus \beta_1^\circ(H_2)$ as

$$\varphi(f) = (f|_{H_1}, f|_{H_2}). \tag{9}$$

Then φ is a surjective algebra isomorphism. Thus there exists a one-to-one correspondence between algebra decompositions of $\beta_1^\circ([0,1])$ and the $\mathcal{H}$ partitions of $[0,1]$.

The interval $[0,1]$ is a second-category topological space, and H_1 and H_2 are countable unions of closed sets, therefore one of them has nonempty interior. Suppose the interior of H_1 is not empty. Then there exists a closed interval $[a,b] \subseteq H_1$. Clearly, we have

$$\beta_1^\circ([0,1]) \cong \beta_1^\circ([a,b]). \tag{10}$$

On the other hand, there exists an $\mathcal{H}$ set, H_3 in $[0,1]$ disjoint from $[a,b]$ such that

$$H_1 = [a,b] \cup H_3. \tag{11}$$

Therefore,

$$\beta_1^\circ(H_1) = \beta_1^\circ([a,b]) \oplus \beta_1^\circ(H_3). \tag{12}$$

Thus $\beta_1^\circ([0,1])$ is complemented in $\beta_1^\circ(H_1)$.

Now we prove the second assertion. Let C be a complemented subalgebra of $\beta_1^\circ([0,1])$. Therefore, there exists an $\mathcal{H}$ set, H in $[0,1]$ such that $C = \beta_1^\circ(H)$. We define

$$\begin{gathered} P : \beta_1^\circ([0,1]) \longrightarrow \beta_1^\circ([0,1]), \\ P(f) = f|_H. \end{gathered} \tag{13}$$

Let $H^c = [0,1] - H$ and $f \in \beta_1^\circ([0,1])$. We have $f = f|_H + f|_{H^c}$. Thus,

$$\|f\| = \max(\|f|_H\|, \|f|_{H^c}\|) \geq \|f|_H\|. \tag{14}$$

Therefore, $\|P\| \leq 1$. It follows that $\|P\| = 1$ since the norm of a projection is always equal to or greater than 1. If $f \in \beta_1^\circ([0,1])$ and $f \geq 0$, then $f|_H \geq 0$, and therefore, P is positive. Also, it is obvious that P is multiplicative. The proof is now complete. □

The finite and small Baire classes have been studied by many people (e.g., [2, 7]). We begin by recalling the definition of the index β. Suppose that H is an $\mathscr{H}$ set in $[0,1]$, and f is a real-valued function whose domain is H. For any $\epsilon > 0$, let $H^0(f,\epsilon) = H$. If $H^\alpha(f,\epsilon)$ is defined for some countable ordinal α, let $H^{\alpha+1}(f,\epsilon)$ be the set of all those $x \in H^\alpha(f,\epsilon)$ such that for every open U containing x, there are two points x_1 and x_2 in $U \cap H^\alpha(f,\epsilon)$ with $|f(x_1) - f(x_2)| \geq \epsilon$. For a countable limit ordinal α, we let

$$H^\alpha(f,\epsilon) = \cap_{\alpha'<\alpha} H^{\alpha'}(f,\epsilon). \tag{15}$$

The index $\beta_H(f,\epsilon)$ is taken to be the least α with $H^\alpha(f,\epsilon) = \emptyset$ if such α exists, and ω_1 otherwise. The oscillation index of f is

$$\beta_H(f) = \sup\{\beta_H(f,\epsilon) : \epsilon > 0\}. \tag{16}$$

It is known that $f : H \to \mathbb{R}$ is Baire-1 if and only if $\beta_H(f) < \omega_1$ [7]. We define the set of bounded functions of finite Baire index (resp., small Baire class ξ for each countable ordinal ξ) as

$$\mathscr{B}_1^F(H) = \{f \in \beta_1^\circ(H) : \beta_H(f) < \infty\}, \qquad (\mathscr{B}_1^\xi(H) = \{f \in \beta_1^\circ(H) : \beta_H(f) \leq \omega^\xi\}). \tag{17}$$

The set of bounded functions of finite Baire index, $\mathscr{B}_1^F$ (resp., small Baire class ξ, $\mathscr{B}_1^\xi(H)$), is a Banach algebra (with sup-norm). It is obvious that if $f : H_1 \to \mathbb{R}$ is a Baire-1 function, $H_2 \subseteq H_1 \subseteq [0,1]$ ($H_1, H_2 \in \mathscr{H}$) and $g = f|_{H_2}$, then $\beta_{H_2}(g) \leq \beta_{H_1}(f)$. Therefore, if $f \in \mathscr{B}_1^F$ (resp., $f \in \mathscr{B}_1^\xi(H_1)$), then $g \in \mathscr{B}_1^F$ (resp., $g \in \mathscr{B}_1^\xi(H_2)$ (but the converse is not true). So we have the following.

Remark 2. The previous theorem is also valid for the set of bounded functions of finite Baire index (resp., small Baire class ξ).

It may happen that for two $\mathscr{H}$ sets, H_1 and H_2 ($\subseteq [0,1]$ such that $H_2 = [0,1] - H_1$), and $f_1 \in \mathscr{B}_1^F(H_1)$ and $f_2 \in \mathscr{B}_1^F(H_2)$, $f = f_1 + f_2$ does not belong to $\mathscr{B}_1^F([0,1])$. Suppose that H_1 is the Cantor set C in the interval $[0,1]$ and $H_2 = [0,1] - C$. Let $f_i = i\chi_{H_i}$ ($i = 1,2$), and $f = f_1 + f_2$. It is obvious that $f \notin \mathscr{B}_1^F([0,1])$. Thus,

$$\mathscr{B}_1^F([0,1]) \subsetneq \mathscr{B}_1^F(H_1) \oplus \mathscr{B}_1^F(H_2). \tag{18}$$

But if M and N are two complementary subalgebras of $\mathscr{B}_1^\xi([0,1])$, then there exist suitable $\mathscr{H}$ sets, H_1 and H_2, such that

$$M = \mathscr{B}_1^F(H_1), \qquad N = \mathscr{B}_1^F(H_2). \tag{19}$$

By the above argument and using the epimorphism

$$\begin{gathered} \Theta : \mathscr{B}_1^F([0,1]) \longrightarrow \mathscr{B}_1^F(C), \\ \Theta(f) = f|_C, \end{gathered} \tag{20}$$

we see that there exists a noncomplemented subalgebra $\mathcal{D}$ such that

$$\frac{\mathcal{B}_1^F([0,1])}{\mathcal{D}} \cong \mathcal{B}_1^F(C), \tag{21}$$

and $\mathcal{D}$ is not of the form $\mathcal{B}_1^F(H)$ for any $\mathcal{H}$ set H. (The algebra homomorphism Θ is onto by Tietze extension theorem [8, Theorem 3.6].)

It has been proved that for real compact spaces X and Y, a linear isometry between $\beta_1^\circ(X)$ and $\beta_1^\circ(Y)$ induces an algebra (a ring) isometry [6]. Is this true for linear complemented subspaces of $\beta_1^\circ([0,1])$? If the answer to the above question is positive, then it should be easy to prove an analogous theorem for linear complemented subspaces of $\beta_1^\circ([0,1])$.

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