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*Research Article*

# Existence of Fixed Point Results in *G*-Metric Spaces

**Zead Mustafa,[1] Wasfi Shatanawi,[1] and Malik Bataineh[2]**

[1] *Department of Mathematics, The Hashemite University, P.O. Box 150459, Zarqa 13115, Jordan*
[2] *Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan*

Correspondence should be addressed to Zead Mustafa, zmagablh@hu.edu.jo

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The purpose of this paper is to prove the existence of fixed points of contractive mapping defined on $G$-metric space where the completeness is replaced with weaker conditions. Moreover, we showed that these conditions do not guarantee the completeness of $G$-metric spaces.



## 1. Introduction

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models, and economic theories.

Different mathematicians tried to generalize the usual notion of metric space $(X, d)$ such as Gähler [1, 2] and Dhage [3–5] to extend known metric space theorems in more general setting, but different authors proved that these attempts are unvalid (for detail see [6–8]).

In 2005, Mustafa and Sims introduced a new structure of generalized metric spaces (see [9]), which are called $G$-metric spaces as generalization of metric space $(X, d)$, to develop and introduce a new fixed point theory for various mappings in this new structure. The $G$-metric space is as follows.

*Definition 1.1* (see [9]). Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying the following:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$, (G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$, (symmetry in all three variables),

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function $G$ is called a *generalized metric*, or, more specifically a *G-metric* on $X$, and the pair $(X,G)$ is a *G-metric space*.

Clearly these properties are satisfied when $G(x,y,z)$ is the perimeter of the triangle with vertices at $x$, $y$, and $z$ in $\mathbb{R}^2$; moreover taking $a$ in the interior of the triangle shows that (G5) is the best possible.

If $(X,d)$ is an ordinary metric space, then $(X,d)$ can define $G$-metrics on $X$ by

($E_s$) $G_s(d)(x,y,z) = d(x,y) + d(y,z) + d(x,z)$,

($E_m$) $G_m(d)(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$.

**Proposition 1.2** (see [9]). *Let $(X,G)$ be a G-metric space. Then for any $x,y,z$, and $a \in X$, it follows that*

(1) *if $G(x,y,z) = 0$, then $x = y = z$,*

(2) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$,

(3) $G(x,y,y) \le 2G(y,x,x)$,

(4) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$,

(5) $G(x,y,z) \le (2/3)(G(x,y,a) + G(x,a,z) + G(a,y,z))$,

(6) $G(x,y,z) \le (G(x,a,a) + G(y,a,a) + G(z,a,a))$.

**Proposition 1.3** (see [9]). *Every G-metric space $(X,G)$ will define a metric space $(X,d_G)$ by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \quad \forall x,y \in X. \tag{1.1}$$

*Definition 1.4* (see [9]). Let $(X,G)$ be a $G$-metric space. Then for $x_0 \in X$, $r > 0$, the *G-ball with center $x_0$ and radius $r$* is

$$B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}. \tag{1.2}$$

**Proposition 1.5** (see [9]). *Let $(X,G)$ be a G-metric space. Then for any $x_0 \in X$ and $r > 0$, one has*

(1) *if $G(x_0,x,y) < r$, then $x,y \in B_G(x_0,r)$,*

(2) *if $y \in B_G(x_0,r)$, then there exists a $\delta > 0$ such that $B_G(y,\delta) \subseteq B(x_0,r)$.*

*Proof.* (1) follows directly from (G3), while (2) follows from (G5) with $\delta = r - G(x_0,y,y)$. □

It follows from (2) of the above proposition that the family of all $G$-balls, $\mathcal{B} = \{B_G(x,r) : x \in X,\ r > 0\}$, is the base of a topology $\tau(G)$ on $X$, the *G-metric topology*.

*Definition 1.6* (see [9]). Let $(X,G)$ be a $G$-metric space, let $(x_n)$ be sequence of points of $X$, a point $x \in X$ is said to be the *limit* of the sequence $(x_n)$ if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, and we say that the sequence $(x_n)$ is *G-convergent* to $x$.

Thus, if $x_n \xrightarrow{(G)} 0$, in a $G$-metric space $(X,G)$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \ge N$, (through this paper we mean by $\mathbb{N}$ the set of all natural numbers).

**Proposition 1.7** (see [9]). *Let $(X, G)$ be a G-metric space. The sequence $(x_n) \subseteq X$ is G-convergent to $x$ if it converges to $x$ in the G-metric topology, $\tau(G)$.*

**Proposition 1.8** (see [9]). *Let $(X, G)$ be a G-metric space. Then for a sequence $(x_n) \subseteq X$ and a point $x \in X$ the following are equivalent*

(1) $(x_n)$ *is G-convergent to* $x$,

(2) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$,

(3) $G(x_n, x, x) \to 0$, *as* $n \to \infty$,

(4) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

*Definition 1.9* (see [9]). Let $(X, G)$ and $(X', G')$ be $G$-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then $f$ is said to be *G-continuous at a point* $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function $f$ is $G$-continuous at $X$ if and only if it is $G$-continuous at all $a \in X$.

**Proposition 1.10** (see [9]). *Let $(X, G)$, $(X', G')$ be G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at $x$; that is, whenever $(x_n)$ is G-convergent to xone has $(f(x_n))$ is G-convergent to $f(x)$.*

**Proposition 1.11** (see [9]). *Let $(X, G)$ be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

*Definition 1.12* (see [9]). Let $(X, G)$ be a $G$-metric space. Then the sequence $(x_n) \subseteq X$ is said to be *G-Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

*Definition 1.13* (see [9]). A $G$-metric space $(X, G)$ is said to be *G-complete* (or complete $G$-metric space) if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $(X, G)$.

## 2. The Main Results

In this section we will prove several theorems in each of which we have omitted the completeness property of $G$-metric space and we have obtained the same conclusion as in complete $G$-metric space, but with assumed sufficient conditions.

**Theorem 2.1.** *Let $(X, G)$ be a G-metric space and let $T : X \to X$ be a mapping such that $T$ satisfies that*

(A1) $G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz)$ *for all* $x, y, z \in X$ *where* $0 < a + b + c < 1$,

(A2) $T$ *is G-continuous at a point* $u \in X$,

(A3) *there is* $x \in X$; $\{T^n(x)\}$ *has a subsequence* $\{T^{ni}(x)\}$ *G-converges to* $u$. *Then* $u$ *is a unique fixed point (i.e.,* $Tu = u$*).*

*Proof.* $G$-continuity of $T$ at $u$ implies that $\{T^{ni+1}(x)\}$ $G$-convergent to $T(u)$. Suppose $T(u) \neq u$, consider the two $G$-open balls $B_1 = B(u, \epsilon)$ and $B_2 = B(Tu, \epsilon)$ where $\epsilon < (1/6) \min\{G(u, Tu, Tu), G(Tu, u, u)\}$.

Since $T^{ni}(x) \to u$ and $T^{ni+1}(x) \to Tu$, then there exist $N_1 \in \mathbb{N}$ such that if $i > N_1$ implies $T^{ni}(x) \in B_1$ and $T^{ni+1}(x) \in B_2$. Hence our assumption implies that we must have

$$G\Big(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\Big) > \epsilon, \quad \forall i > N_1. \tag{2.1}$$

On the other hand we have from (A1),

$$\begin{aligned} G\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\Big) \le\ & aG\Big(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\Big) \\ & + bG\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\Big) \\ & + cG\Big(T^{ni+2}(x), T^{ni+3}(x), T^{ni+3}(x)\Big) \end{aligned} \tag{2.2}$$

but, by axioms of $G$-metric (G3), we have

$$G\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\Big) \le G\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\Big), \tag{2.3}$$

$$G\Big(T^{ni+2}(x), T^{ni+2}(x), T^{ni+3}(x)\Big) \le G\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\Big). \tag{2.4}$$

So, from (2.3) and (2.4), we see (2.2) becomes

$$G\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\Big) \le\ qG\Big(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\Big), \tag{2.5}$$

where $q = a/(1-(b+c))$ and $q < 1$, since $0 < a + b + c < 1$.

Hence (2.3) and (2.5) implies that

$$G\Big(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\Big) \le qG\Big(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\Big). \tag{2.6}$$

For $l > j > N_1$ and by repeated application of (2.6) we have

$$\begin{aligned} G\Big(T^{n_l}(x), T^{n_l+1}(x), T^{n_l+1}(x)\Big) &\le qG\Big(T^{n_l-1}(x), T^{n_l}(x), T^{n_l}(x)\Big) \\ &\le q^2 G\Big(T^{n_l-2}(x), T^{n_l-1}(x), T^{n_l-1}(x)\Big) \\ &\le \cdots \le q^{n_l-n_j} G\Big(T^{n_j}(x), T^{n_j+1}(x), T^{n_j+1}(x)\Big). \end{aligned} \tag{2.7}$$

So, as $l \to \infty$ we have $\lim G(T^{n_l}(x), T^{n_l+1}(x), T^{n_l+1}(x)) \le 0$ which contradict (2.1), hence $Tu = u$.

Suppose there is $v \in X; Tv = v$, then from (A1), we have

$$G(u,v,v) = G(Tu,Tv,Tv) \leq aG(u,Tu,Tu) + (b+c)G(v,Tv,Tv) = 0. \tag{2.8}$$

This prove the uniqueness of $u$. □

In [10] we have proved the following theorem.

**Theorem 2.2** (see [10]). *Let $(X,G)$ be a complete $G$-metric space and let $T : X \to X$ be a mapping satisfies the following condition for all $x,y,z \in X$:*

$$\begin{aligned} &G(T(x),T(y),T(z)) \\ &\quad \leq aG(x,T(x),T(x)) + bG(y,T(y),T(y)) + cG(z,T(z),T(z)) + dG(x,y,z), \end{aligned} \tag{2.9}$$

*where $0 \leq a+b+c+d < 1$, then $T$ has a unique fixed point, say $u$, and $T$ is $G$-continuous at $u$.*

We see that if we take $d = 0$, the following theorem becomes a direct result.

**Theorem 2.3.** *Let $(X,G)$ be complete $G$-metric space and let $T : X \to X$ be a mapping satisfies for all $x,y,z \in X$*

$$G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz), \tag{2.10}$$

*where $0 < a+b+c < 1$, then $T$ has a unique fixed point, say $u$, and $T$ is $G$-continuous at $u$.*

If we compare Theorem 2.3 with Theorem 2.1, we see that in Theorem 2.1 we have omitted the completeness property of the $G$-metric space and instead we have assumed conditions (2) and (3). However, the following examples support that conditions (2) and (3) in Theorem 2.1 do not guarantee the completeness of the $G$-metric space.

*Example 2.4.* Let $X = [0,1)$, $T(x) = x/4$ and $G(x,y,z) = \max\{|x-y|,|y-z|,|x-z|\}$. Then $(X,G)$ is $G$-metric space but not complete, since the sequence $x_n = 1 - 1/n$ is $G$-cauchy which is not $G$-convergent in $(X,G)$. However, conditions (2) and (3) in Theorem 2.1 are satisfied.

**Theorem 2.5.** *Let $(X,G)$ be a $G$-metric space and let $T : X \to X$ be a $G$-continuous mapping satisfies the following conditions:*

(B1) *$G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)$ for all $x,y,z \in M$, where $M$ is an every where dense subset of $X$ (with respect the topology of $G$-metric convergence) and $0 < k < 1/6$,*

(B2) *there is $x \in X$ such that $\{T^n(x)\} \to x_o$. Then $x_o$ is unique fixed point.*

*Proof.* The proof will follow from Theorem 2.1, if we can show that condition (A1) in Theorem 2.1 holds for any $x,y,z \in X$.

Let $x$, $y$, $z$ be any elements of $X$.

*Case 1.* If $x, y, z \in X \setminus M$, let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be a sequences in $M$ such that $y_n \to y$, $x_n \to x$ and $z_n \to z$. From (G5) we have

$$G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty), \tag{2.11}$$

also

$$G(Tz, Ty, Ty) \leq G(Tz, Tz_n, Tz_n) + G(Tz_n, Ty_n, Ty_n) + G(Ty_n, Ty, Ty) \tag{2.12}$$

and by (B1), we have

$$G(Tz_n, Ty_n, Ty_n) \leq k\{G(z_n, Tz_n, Tz_n) + 2G(y_n, Ty_n, Ty_n)\}, \tag{2.13}$$

again by (G5) we have

$$\begin{aligned} G(z_n, Tz_n, Tz_n) &\leq G(z_n, z, z) + G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n), \\ G(y_n, Ty_n, Ty_n) &\leq G(y_n, y, y) + G(y, Ty, Ty) + G(Ty, Ty_n, Ty_n). \end{aligned} \tag{2.14}$$

So, from (2.13) and (2.14) we see that (2.12) becomes

$$\begin{aligned} &G(Tz, Ty, Ty) \\ &\quad \leq (1+k)G(Tz, Tz_n, Tz_n) + G(Ty_n, Ty, Ty) + kG(z_n, z, z) \\ &\qquad + 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(z, Tz, Tz) + 2kG(y, Ty, Ty), \end{aligned} \tag{2.15}$$

by the same argument we deduce that

$$\begin{aligned} &G(Tx, Ty, Ty) \\ &\quad \leq (1+k)G(Tx, Tx_n, Tx_n) + G(Ty_n, Ty, Ty) + kG(x_n, x, x) \\ &\qquad + 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(x, Tx, Tx) + 2kG(y, Ty, Ty). \end{aligned} \tag{2.16}$$

Hence, by (2.15) and (2.16), we have

$$\begin{aligned} G(Tx, Ty, Tz) &\leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty) \\ &\leq \{(1+k)G(Tx, Tx_n, Tx_n) + G(Ty_n, Ty, Ty) + kG(x_n, x, x) \\ &\quad + 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(x, Tx, Tx) + 2kG(y, Ty, Ty)\} \\ &\quad + \{(1+k)G(Tz, Tz_n, Tz_n) + G(Ty_n, Ty, Ty) + kG(z_n, z, z) \\ &\quad + 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(z, Tz, Tz) + 2kG(y, Ty, Ty)\}. \end{aligned} \tag{2.17}$$

Now letting $n \to \infty$ in the above inequality and using the fact that $T$ is $G$-continuous we get

$$G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx) + 4G(y,Ty,Ty) + G(z,Tz,Tz)\}. \tag{2.18}$$

*Case 2.* If $x, y \in M$ and $z \in X \setminus M$, let $\{z_n\}$ be a sequence in $M$ such that $z_n \to z$, then by (G5), we have

$$G(Tx,Ty,Tz) \leq G(Tx,Ty,Ty) + G(Tz,Ty,Ty) \tag{2.19}$$

but by (B1), we have

$$G(Tx,Ty,Ty) \leq k\{G(x,Tx,Tx) + 2G(y,Ty,Ty)\}, \tag{2.20}$$

and by (G5), we have

$$G(Tz,Ty,Ty) \leq G(Tz,Tz_n,Tz_n) + G(Tz_n,Ty,Ty). \tag{2.21}$$

Again by (B1), we have

$$G(Tz_n,Ty,Ty) \leq k\{G(z_n,Tz_n,Tz_n) + 2G(y,Ty,Ty)\}. \tag{2.22}$$

Also, by (G5), we have

$$G(z_n,Tz_n,Tz_n) \leq G(z_n,z,z) + G(z,Tz,Tz) + G(Tz,Tz_n,Tz_n). \tag{2.23}$$

So, from (2.21), (2.22), and (2.23), we see that (2.19) becomes

$$\begin{aligned} &G(Tx,Ty,Tz) \\ &\quad\leq k\{G(x,Tx,Tx) + 2G(y,Ty,Ty) + G(z_n,z,z) + G(z,Tz,Tz) + G(Tz,Tz_n,Tz_n)\} \\ &\qquad+ G(Tz,Tz_n,Tz_n) + 2kG(y,Ty,Ty). \end{aligned} \tag{2.24}$$

Now letting $n \to \infty$ in the above inequality, we get

$$G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx) + 4G(y,Ty,Ty) + G(z,Tz,Tz)\}. \tag{2.25}$$

*Case 3.* If $y \in M$ and $x, z \in X \setminus M$, let $\{x_n\}$ and $\{z_n\}$ be a sequences in $M$ such that $x_n \to x$ and $z_n \to z$, but by (G5), we have

$$G(Tx,Ty,Tz) \leq G(Tx,Ty,Ty) + G(Tz,Ty,Ty), \tag{2.26}$$

$$G(Tx,Ty,Ty) \leq G(Tx,Tx_n,Tx_n) + G(Tx_n,Ty,Ty), \tag{2.27}$$

also, from (B1), we have

$$G(Tx_n, Ty, Ty) \le k\{G(x_n, Tx_n, Tx_n) + 2G(y, Ty, Ty)\}, \tag{2.28}$$

and from (G5), we have

$$G(x_n, Tx_n, Tx_n) \le G(x_n, x, x) + G(x, Tx, Tx) + G(Tx, Tx_n, Tx_n). \tag{2.29}$$

So, by (2.28) and (2.29), we have

$$G(Tx_n, Ty, Ty) \le 2kG(y, Ty, Ty) + kG(x_n, x, x) + kG(x, Tx, Tx) + kG(Tx, Tx_n, Tx_n), \tag{2.30}$$

then from (2.27) and (2.30) we have

$$G(Tx, Ty, Ty) \le kG(x_n, x, x) + kG(x, Tx, Tx) + (1+k)G(Tx, Tx_n, Tx_n) + 2kG(y, Ty, Ty). \tag{2.31}$$

By the same argument we deduce that

$$G(Tz, Ty, Ty) \le kG(z_n, z, z) + kG(z, Tz, Tz) + (1+k)G(Tz, Tz_n, Tz_n) + 2kG(y, Ty, Ty). \tag{2.32}$$

Then, from (2.31) and (2.32), we see (2.26) becomes

$$\begin{aligned} G(Tx, Ty, Tz) &\le G(Tx, Ty, Ty) + G(Tz, Ty, Ty) \\ &\le kG(x_n, x, x) + kG(x, Tx, Tx) + kG(Tx, Tx_n, Tx_n) + 2kG(y, Ty, Ty) \\ &\quad + kG(z_n, z, z) + kG(z, Tz, Tz) + kG(Tz, Tz_n, Tz_n) + 2kG(y, Ty, Ty). \end{aligned} \tag{2.33}$$

Now letting $n \to \infty$ in the above inequality and using the fact that $T$ is $G$-continuous we get

$$G(Tx, Ty, Tz) \le k\{G(x, Tx, Tx) + 4G(y, Ty, Ty) + G(z, Tz, Tz)\}. \tag{2.34}$$

So, in all cases we have for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \le aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz), \tag{2.35}$$

where $a = k$, $b = 4k$, $c = k$, and $a + b + c < 1$ since $0 < k < 1/6$, then by Theorem 2.1, $T$ has a unique fixed point. □

**Corollary 2.6.** *Let $(X, G)$ be G-metric space and let $T : X \to X$ be a mapping such that $T$ satisfies that*

(C1) *$G(Tx, Ty, Ty) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty)$ for all $x, y \in X$ where $0 < a + b < 1$,*

(C2) *$T$ is G-continuous at a point $u \in X$,*

(C3) *there is $x \in X$; $\{T^n(x)\}$ has a subsequence $\{T^{ni}(x)\}$ G-converges to $u$. Then $u$ is a unique fixed point.*

*Proof.* Let $z = y$ in condition (A1), then we see that every mapping satisfies condition (C1) will satisfy condition (A1), so the proof follows from Theorem 2.1. □

**Corollary 2.7.** *Let $(X, G)$ be G-metric space and let $T : X \to X$ be a G-continuous mapping satisfies that*

(D1) *$G(Tx, Ty, Ty) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty)\}$ for all $x, y \in M$ where $M$ is an every where dense subset of $X$ (with respect the topology of G-metric convergence) and $0 < k < 1/6$,*

(D2) *there is $x \in X$ such that $\{T^n(x)\} \to x_o$. Then $x_o$ is unique fixed point.*

*Proof.* Let $z = y$ in condition (B1), then we see that every mapping satisfies condition (D1) will satisfy condition (B1), so the proof follows from Theorem 2.5. □

**Corollary 2.8.** *Let $(X, G)$ be G-metric space and let $T : X \to X$ be a mapping such that $T$ satisfies that*

(E1) *$G(Tx, Ty, Ty) \leq kG(x, y, y)$ for all $x, y \in X$ where $0 < k < 1/4$,*

(E2) *$T$ is G-continuous at a point $u \in X$,*

(E3) *there is $x \in X$; $\{T^n(x)\}$ has a subsequence $\{T^{ni}(x)\}$ G-converges to $u$. Then $u$ is a unique fixed point.*

*Proof.* By axioms of G-metric (G5), we have

$$\begin{gathered} G(x, y, y) \leq G(x, Tx, Tx) + G(Tx, Ty, Ty) + G(Ty, y, y), \\ G(Ty, y, y) \leq 2G(y, Ty, Ty), \end{gathered} \tag{2.36}$$

so, from (2.36), we see that (E1) becomes

$$G(Tx, Ty, Ty) \leq kG(x, y, y) \leq kG(x, Tx, Tx) + kG(Tx, Ty, Ty) + 2kG(y, Ty, Ty), \tag{2.37}$$

then $T$ will satisfy the following condition

$$G(Tx, Ty, Ty) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) \tag{2.38}$$

for all $x, y \in X$, where $a = k/(1-k)$, $b = 2k/(1-k)$, and $a + b < 1$, since $k < 1/4$.

So, condition (C1) is satisfied and the proof follows from Corollary 2.6. □

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