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THE NUMBER OF CLONES CONTAINING AN UNARY FUNCTION

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Abstract

It is known that for arbitrary transformation monoid M on a set A the clones C on A with $C^{(1)} = M$ form an interval $Int(M)$ in the clone lattice. The problem is [5]: for which transformation monoids M on $E_k, k > 2$,

(a) $Int(M)$ is finite,
(b) $|Int(M)| = 2^{\aleph_0}$?

In this paper we show that there are continuum of clones containing a Picar function ([3], $theorem\ 9, p.54$) and $|Int(M)| = 2^{\aleph_0}$ where M is a given special transformation monoid.

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1. Notation and Preliminaries

Let A be a set. By an operation we will always mean a finitary, non-nullary operation. For integers $n \geq 1$ and $1 \leq i \leq n$, the i-th n-ary projection on A is the operation defined by

$$e_{n,i}(a_1, \ldots, a_n) = a_i \ \ for \ \ all \ \ a_1, \ldots, a_n \in A.$$

If f is an n-ary and $g_1, \ldots, g_n$ are k-ary operations on A, then we define a k-ary operation $f(g_1, \ldots, g_n)$ on A, called the superposition of $f, g_1, \ldots, g_n$, as follows:

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_k) = f(g_1(a_1, \ldots, a_k), \ldots, g_n(a_1, \ldots, a_k))$$

for all $a_1, \ldots a_k \in A$. A set of operations on a fixed set A is said to be a clone on A iff it contains the projections and is closed under superposition.

The clones on A form a complete lattice $Lat(a)$ in which the least element is the clone of all projections and the greatest element is the clone of all operations on A. For an arbitrary set F of operations on A there exists the least clone containing F. This clone is called the clone generated by F, and will be denoted by $\langle F \rangle_{\mathrm{CL}}$. Instead of $\langle \{f\} \rangle_{\mathrm{CL}}$ we will write simply $\langle f \rangle_{\mathrm{CL}}$. For a clone C and $n \geq 1$ we denote by $C^{(n)}$ the set of n-ary operations from C.

Given a transformation monoid (that is a monoid of unary operations) M on a set A, the problem is to describe all clones C with $C^{(1)} = M$. For arbitrary transformation monoid M on a set A, the clones C on A with $C^{(1)} = M$ form an interval in the clone lattice. This interval is denoted by $Int(M)$.

2. The Number of Clones Containing a Unary Function

Let $A = \{0, 1, \ldots, k-1\}$ and $h(x) = \begin{cases} 1 & , \ \ if \ \ x = 0 \\ x & , \ \ otherwise. \end{cases}$.

For $|A| = 2$ it follows from Post's lattice that there are finitely many clones containing h.

Theorem 2.1. *The cardinality of the set of clones containing h on a finite set A is the continuum for $|A| > 2$.*

Proof. We define a countable set of functions F in such a way that for each $f \in F$

$$f \notin \langle (F \setminus \{f\}) \cup \{h\} \rangle_{\mathrm{CL}}.$$

This implies that for each $G, H \subseteq F$, if $G \neq H$ then $\langle G \cup \{h\} \rangle_{\mathrm{CL}} \neq \langle H \cup \{h\} \rangle_{\mathrm{CL}}$. In this way we get a set of distinct clones of continuum cardinality each containing h. On the other hand, it is known that there are no more than continuum clones containing h.

In the following, $\mathbf{x} = (x^1, \ldots, x^m)$ and $h(\mathbf{x}) = \mathbf{y}$ means that $h(x^i) = y^i$ for $i \in \{1, \ldots m\}$.

For $m > 2$ let us define the $m-$ary function f_m as follows:

$$f_m(\mathbf{x}) = \begin{cases} 0 & if \ \ (|\{j : x^j = 2\}| = 1 \ and \ |\{j : x^j = 0\}| = m - 1) \\ 1 & otherwise \end{cases}$$

and let F be the set of all these functions:

$$F = \bigcup_{m>2} \{f_m\}.$$

For each $m > 2$ we define relation ϱ_m which is preserved by h, and we are going to prove the independence of the set F by showing that f_m does not preserve ϱ_m and all the functions $f_i, i \neq m$, do preserve ϱ_m.

Let us define the following relations $\varrho_m \subseteq A^m$ on A for $m > 2$:

$$\varrho_m = B_m \cup C_m \cup D_m$$

where:

$$B_m = \{\mathbf{b} \in A^m : \{|j : b^j = 0\}| = m - 1 \wedge |\{j : b^j = 2\}| = 1\}|\},$$

$$C_m = \{\mathbf{c} \in A^m : \{|j : c^j = 1\}| = m - 1 \wedge |\{j : c^j = 2\}| = 1\}|\} \ and$$

$$D_m = \{0,1\}^m \setminus \{(\underbrace{0, \ldots, 0}_{m})\}.$$

It is clear that h preserves ϱ_m since:

$$h(\mathbf{b}) \in C_m \ \ if \ \ \mathbf{b} \in B_m;$$

$$h(\mathbf{c}) = \mathbf{c} \;\; if \;\; \mathbf{c} \in C_m;$$
$$h(\mathbf{d}) = (1,1,...,1) \in D_m \;\; if \;\; \mathbf{d} \in D_m.$$

If we take $\mathbf{x_1}, \ldots, \mathbf{x_m} \in B_m \subseteq \varrho_m$ with $x_i^i = 2$, then

$(f_m(x_1^1, \ldots, x_m^1), \ldots, \; f_m(x_1^m, \ldots, x_m^m)) = (0, \ldots, 0) \notin \varrho_m$. So, $f_m \notin Pol\varrho_m$.

Since $Imf_i = \{0,1\}$ and $i \neq m$ it is sufficient to prove that there are no vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i \in \varrho_m$ such that

$$(f_i(x_1^1, \ldots, x_i^1), \ldots, f_i(x_1^m, \ldots, x_i^m)) = (0, \ldots, 0).$$

Let us suppose that there are such vectors $\mathbf{x}_1, \ldots, \mathbf{x}_i \in \varrho_m$. Then the vectors, $\mathbf{x}_1, \ldots, \mathbf{x}_i \in B_m$, because in the oposite case it is obvious that

$$(f_i(x_1^1, \ldots, x_i^1), \ldots, f_i(x_1^m, \ldots, x_i^m)) \neq (0, \ldots, 0).$$

If $i > m$ i.e. $\mathbf{x}_p = \mathbf{x}_q$ for $p, q \in \{1, \ldots, i\}$ and $x_p^l = 2$ then

$$(\ldots, f_i(\ldots, x_p^l, \ldots, x_q^l, \ldots), \ldots) = (\ldots, f_i(\ldots, 2, \ldots, 2, \ldots), \ldots) = (\ldots, 1, \ldots).$$

So, $i < m$ and there is a vector $\mathbf{x}_j \in B_m$ such that $\mathbf{x}_j$ does not appear among $\mathbf{x}_1, \ldots, \mathbf{x}_i$. Suppose that $x_j^l = 2$. We have

$$(\ldots, f_i(x_1^l, \ldots, x_m^l), \ldots) = (\ldots, f_i(0, \ldots, 0), \ldots) = (\ldots, 1, \ldots).$$

Contradiction. So, we have shown that the operations $f_i, i \neq m$ preserve the relation ϱ_m. Now we have

$$h \in Pol\varrho_m \;\wedge\; f_m \notin Pol\varrho_m \;\wedge\; f_m \neq f_i \in Pol\varrho_m \quad \text{which implies}$$

$$f_m \notin \langle (F \setminus \{f_m\}) \cup \{h\} \rangle_{\mathrm{CL}} \text{ because } Pol\varrho_m \;\supset\; \langle (F \setminus \{f_m\}) \cup \{h\} \rangle_{\mathrm{CL}}.$$

□

Theorem 2.2. *Let* $g_1(x) = x$ *for all* $x \in \{0,1,2\}$,

$$g_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, g_4 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, g_5 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

If $M = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$, *where* M_i *is the set of all functions whose restriction is* g_i $(i = 1,2,3,4,5)$, *then*

$$|\{C : C^{(1)} = M\}| = 2^{\aleph_0}.$$

Proof. It is easy to verify that if the relation ϱ_m is defined as in the previous proof, then $Pol^{(1)}\varrho_m = M$. □

3. Conclusions

It remains to describe all the clones that contain h for $|A| > 2$.

If $k = 2$, there are 6 transformation monoids and $\aleph_0$ clones on A. As the clone lattice is fully known, is easy to determine the interval $Int(M)$.

For a finite set A with $|A| \geq 3$ the clone lattice $Lat(A)$ has the continuum elements and little is known about the structure of the lattices in this case. Therefore, the solution of the problem of A. Szendrei would contribute to a better understanding of the structure of $Lat(A)$.

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