An elementary mathematical theory for
the variability hypothesis

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ABSTRACT. A selectivity theory is proposed to help explain how one
gender of a species might tend to evolve with greater variability than
the other gender. Briefly, the theory says that if one sex is relatively
selective, then more variable subpopulations of the opposite sex will tend
to prevail over those with lesser variability; and conversely, if one sex is
relatively nonselective, then less variable subpopulations of the opposite
sex will tend to prevail over those with greater variability. This theory
makes no assumptions about differences in means between the sexes,
nor does it presume that one sex is selective and the other nonselective.
Two mathematical models are presented: a statistical analysis using
normally distributed fitness values, and a deterministic analysis using
exponentially distributed fitness levels and a standard system of coupled
ordinary differential equations. The theory is applied to the classical
greater male variability hypothesis.

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1. Introduction

The variability hypothesis (VH), also known as the greater male variability hypothesis, asserts that “males are more likely than females to vary from the norm in both physical and mental traits” [Sh99]. The origin of the hypothesis is often traced back to Johann Meckel in the early eighteenth century, and it was used by Charles Darwin to try to explain extreme male secondary sex characteristics in many species. It should be emphasized that the VH applies to all species with male and female sexes, and that it says nothing about differences in the means between males and females of the species.

The variability hypothesis as applied to humans proved highly controversial during the last century and was allegedly in decline. The past fifteen years, however, have seen a resurgence of research on this topic. Johnson et al reviewed scores of articles on the VH as applied to general intelligence in humans and found that the data “substantially support the [variability] hypothesis” [JCD08, p. 529]. Lehre et al performed a retrospective statistical analysis of human intrasex variability and concluded that “human greater male intrasex variability is not limited to intelligence test scores, and [the data] suggest that generally greater intrasex variability among males is a fundamental aspect of the differences between sexes” [LLLD08, p. 204].

A resurgence of controversy came after the VH was linked to the forced resignation of Harvard President Larry Summers and the firing of Google engineer James Damore [SH17a]. Prompted by those incidents, Stevens and Haidt recently conducted a meta-analysis of VH studies in humans [SH17b] that includes both findings that support the hypothesis and emphasize potential biological factors for the VH, and findings that contradict the VH and emphasize cultural factors. That study, which has online links to numerous research papers and graphs, concludes “On average, male variability is greater than female variability on a variety of measures of cognitive ability, personality traits, and interests. This means men are more likely to be found at both the low and high end of these distributions”.

Thus, despite the controversies, there appears to be mounting evidence supporting the variability phenomenon for many traits (e.g., see also [DG14] and [Hi17]), and that raises a wide variety of questions. For example, there is the natural underlying question of why there might be such gender differences in variability within a single species. In their summary of Halpern et al’s paper, Stevens and Haidt emphasized the statement that “the reasons why males are often more variable remain elusive” [SH17b], suggesting that this is still an important unsolved issue.

Several other studies, however, have proposed a general evolutionary biological principle of sexual selection that may help explain the VH, namely,
“Greater male variance is among the possible consequences arising from the fact that males ultimately face fiercer intrasexual competition for access to mates than do females” [AP06, p. 46]

and

“the sex that experiences more intense intrasexual competition or more vetting by the other sex will tend to show greater within-sex variation on many traits” [Ge07, p. 176].

To date, these notions of variability and vetting have remained vague, and it is the goal of this paper to provide an elementary mathematical theory where both are clearly defined (e.g., standard deviation for variability does not work), and where not only is this general principle true — a sex that is subject to intense vetting tends toward greater variability — but the converse is also true — a sex that is subject to relatively little vetting tends toward lesser variability, independent of the mean and variance of the opposite sex. Thus, in the constraints of this cross-species model, this offers two independent explanations for the appearance of VH.

Any reasonable explanation of a secondary sex characteristic, such as greater male variability, must be grounded in accepted primary sex characteristics, and such is the case here. One of the fundamental ways that males and females differ is in their roles in reproduction, and a “basic cross-species pattern is that the sex with the slower potential rate of reproduction (typically females, because of gestation time) invests more in parenting, [and] is selective in mate choices” and the “sex with the faster potential rate of reproduction (typically males) invests less in parenting, [and] is less selective in mate choices” [Ge07, p. 175].

2. Selectivity and variability

What does “more selective” or “less selective” mean in this context? To quantify and simplify the concept of acceptability by the opposite sex, many biologists and evolutionary game theorists often use the term fitness for a single real number assigned to each individual that reflects its attractiveness to the opposite sex, with a larger number being preferable to a smaller one. In some sense this numerical value describes how good its particular genotype is at leaving offspring, and represents an individual’s level of reproductive success relative to some baseline level [Sa09]. In reality, of course, attraction and acceptability are very relative and individual concepts, and in that sense this single numerical fitness value may simply reflect an average acceptability over many attributes. The “fitness” of individuals within sexes varies, and its (normalized) distribution is a probability distribution.

To address the main tenet of the VH, the notion of one distribution or population being more (or less) variable than another must first be clarified. There are many possibilities for such a definition, for instance via standard
deviations or first absolute central moments, but those can be quite misleading in the VH context since a single outlier can dramatically alter the value of such statistics. On the other hand, basic comparisons of the survival functions of the two fitness distributions leads to a natural notion of greater variability.

Let $P_1$ and $P_2$ be probability measures with identical medians $m$. Say that $P_1$ is more variable than $P_2$ if the proportions of $P_1$ both above every upper (larger than median) threshold and the proportions below every lower threshold level are greater than those for $P_2$. That is, both upper and lower tails of the $P_1$ distribution are heavier than those of the $P_2$ distribution, for all thresholds. In other words, $P_1$ is more variable than $P_2$ if

$$S_{P_1}(x) > S_{P_2}(x) \text{ for all } x > m \quad \text{and} \quad S_{P_1}(x) < S_{P_2}(x) \text{ for all } x < m,$$

where $S_P$ denotes the survival function for $P$, i.e., $S_P(x)$ is the proportion of a population with distribution $P$ that is above the threshold $x$.

Note that this definition does not require finite standard deviations or symmetry of the distributions, although the examples provided below have both. Similar conclusions may be drawn about one-sided variability, and this is left to the interested reader. Some assumption on two distributions (of the same sex) having comparable average attributes is clearly necessary to be able to draw any useful conclusions in the selectivity context; the assumption of identical medians used here is one natural candidate.

For two normal (Gaussian, or bell-shaped) distributions with the same average value, comparison of variability in this sense is straightforward.

**Example 1.** Let $N(\mu, \sigma^2)$ denote a normal distribution with mean $\mu$ and standard deviation $\sigma$. It can be shown using elementary properties of the gaussian distribution that $N(\mu, \sigma_1^2)$ is more variable than $N(\mu, \sigma_2^2)$ if and only if $\sigma_1^2 > \sigma_2^2$ (see Proposition 6.1 below). This is illustrated in Figure 1, where the area under the more variable red curve to the right of the green line is larger than the area under the blue curve to the right of the green line, even though the blue curve is above the red curve in places. Similarly, for every cutoff below the common mean value for the two subpopulations, the area under the less variable blue curve to the right of the purple threshold value is larger than the area under the red curve to the right of that value.

3. A variability theory based on selectivity

To introduce the theory below, it will be assumed that for each sex there is an upper proportion $p$ between 0 and 1 of the opposite sex that is acceptable for mating. For example, if $p_A$ is that proportion for sex $A$, then every member of sex $A$ will mate with individual $b$ in $B$ if and only if $b$ is in the top $p_A$ fraction of $B$. If $p_A < 0.5$, then sex $A$ is said to be (relatively) selective, and if $p_A > 0.5$, then $A$ is said to be nonselective. For instance, if $p_A = 0.25$, then sex $A$ is selective, since it will mate only with the top
Figure 1. Variability in normal distributions. The red curve is the distribution of the more variable normal subpopulation $B_1$ in Example 3 below. The blue curve is the distribution of the less variable subpopulation $B_2$. The vertical green line is the threshold cutoff for the opposite sex $A$ so that exactly 25% of the composite $B$ population has fitness level above that point. The vertical purple line is the value so that exactly 75% of the $B$ population has fitness level above that point.

quarter of sex $B$, and if $p_A = 0.75$, then sex $A$ is nonselective, since it will mate with any but the bottom quarter of sex $B$.

N.B. Of course this severe condition, as well as the above formal assumptions about fitness levels, are clearly not satisfied in most real life scenarios, and are simply employed here as a starting point for discussion of the general ideas. Similarly, in this simple model it is assumed that the populations are large and mobile so there are always available potential mates of the opposite sex above the threshold cutoff.

Using these definitions of variability and selectivity, it is now straightforward to state the general theory suggested above:

Selectivity theory of variability. In a population with two sexes $A$ and $B$, both of which are needed for reproduction, suppose that sex $A$ is relatively selective, i.e., will mate only with a top tier (less than half) of $B$ candidates. Then among subpopulations of $B$ with comparable average attributes, those with greater variability will tend to prevail over those with lesser variability. Conversely, if $A$ is relatively nonselective, accepting all but a bottom fraction (less than half) of the opposite sex, subpopulations
Note that this theory makes no assumptions about differences in means between the sexes, nor does it presume that one sex is selective and the other nonselective. If both sexes happen to be selective, for instance, then the best evolutionary strategy for each is to tend toward greater variability.

The underlying ideas of this selectivity theory of gender differences in variability can perhaps best be seen through an elementary hypothetical scenario where the subpopulations have distinct fitness levels, and one is trivially more variable than the other. It is assumed here and throughout that offspring are distributed the same way as their parent subpopulation, and that the number of offspring is independent of the fitness value given that it is above the threshold cutoff level.

**Example 2.** Sex $B$ consists of two subpopulations $B_1$ and $B_2$. Half of the fitness values of $B_1$ are uniformly distributed between 1 and 2 and the other half are uniformly distributed between 3 and 4, while all of the fitness values of $B_2$ are uniformly distributed between 2 and 3. Thus $B_1$ is more variable than $B_2$, and they both have the same average fitness.

Suppose first that $B_1$ and $B_2$ are of equal size. Then one quarter of sex $B$ (the lower half of $B_1$) has fitness values between 1 and 2, half of $B$ (all of $B_2$) has fitness between 2 and 3, and one quarter of $B$ (the upper half of $B_1$) has values between 3 and 4. If sex $A$ is relatively selective and will mate only with the top quarter of sex $B$, then all of the next generation will be offspring of the more variable subpopulation $B_1$. On the other hand, if sex $A$ is relatively nonselective and will mate with any but the lower quarter of $B$, then all of the less variable $B_2$ will mate, but only half of the more variable $B_1$ will mate.

Similar conclusions follow if the initial subpopulations are not of equal size. For example, suppose that one third of sex $B$ is the more variable $B_1$ and two thirds is the less variable $B_2$. Then if sex $A$ only mates with the top quarter of $B$, a short calculation shows that two thirds of the next generation will be offspring of $B_1$ and one third will be offspring of $B_2$, thereby reversing their proportions toward the more variable subpopulation $B_1$. On the other hand, if sex $A$ will mate with any but the lower quarter of $B$, then only two ninths of the next generation will be offspring of $B_1$ and seven ninths will be offspring of $B_2$, thereby increasing the proportion of the less variable subpopulation of sex $B$.

Note the asymmetry here in the mating probabilities; some intuition behind why this occurs may perhaps be gained from the observation that the upper tier of the more variable population will always be able to mate, whether the opposite sex is selective or nonselective.
4. A probabilistic model

Suppose that sex $B$ consists of two subpopulations $B_1$ and $B_2$ ($B = B_1 \cup B_2$), of which a proportion $\beta$ between 0 and 1 is of type $B_1$ (and $1 - \beta$ of type $B_2$). Suppose further that the fitness values (to sex $A$) of sex $B$ are normally distributed, i.e., if $X_1$ and $X_2$ are the fitness values of two random individuals chosen from $B_1$ and $B_2$, respectively, then $X_1$ has distribution $N(\mu, \sigma^2_1)$ and $X_2$ has distribution $N(\mu, \sigma^2_2)$, where $\sigma^2_1 > \sigma^2_2$. The assumption of normality for the underlying distributions of fitness is not essential; this is merely an illustrative example, and chosen because of the ubiquity of the normal distribution in many population studies. (Note the key assumption that the average values, i.e., the medians, are the same.)

As was seen in Example 1, $N(\mu, \sigma^2_1)$ is more variable than $N(\mu, \sigma^2_2)$ since $\sigma^2_1 > \sigma^2_2$. From that it can be shown (see Proposition 6.2 below) that, for all $\beta$, if sex $A$ is selective, i.e., if $p_A < 0.5$, then the probability that a random individual from $B_1$ is selected for mating by sex $A$ is greater than the probability that a random individual from $B_2$ is selected; conversely, if sex $A$ is nonselective, then the probability that a random individual from $B_2$ is selected for mating by sex $A$ is greater than the probability that a random individual from $B_1$ is selected.

Example 3. Suppose $X_1 \sim N(100, 4)$, $X_2 \sim N(100, 1)$, and subpopulations $B_1$ and $B_2$ are of equal size. Two typical cases will again be considered: the case where sex $A$ is selective with $p_A = 0.25$ and the case where sex $A$ is nonselective with $p_A = 0.75$ (see Figure 1).

Suppose first that $p_A = 0.25$. Using a special function calculator (since the survival functions of normal distributions are not known in closed form), it can be determined numerically that sex $A$'s threshold fitness cutoff for sex $B$ is $c^* \approx 100.92$, $S_1(c^*) \approx 0.323$, and $S_2(c^*) \approx 0.179$. Thus a random individual from subpopulation $B_1$ has nearly twice the probability of mating than one from the less variable subpopulation $B_2$, as is illustrated in Figure 1 with the areas to the right of the green cutoff.

Now suppose that $p_A = 0.75$. Then it can be determined that the threshold fitness cutoff is $c^* \approx 99.08$, $S_1(c^*) \approx 0.677$, and $S_2(c^*) \approx 0.821$, i.e., a random individual from subpopulation $B_2$ is about one-fifth more likely to be able to mate than one from the more variable subpopulation $B_1$. This is illustrated in Figure 1 with the areas to the right of the purple cutoff.

Here again, note the asymmetry in that the selective case is more extreme than the nonselective case, as was seen in Example 1.

5. A deterministic model

In this setting sex $B$ consists of two growing subpopulations $B_1$ and $B_2$, i.e., $B = B_1 \cup B_2$, whose sizes at time $t$ are $x_1(t)$ and $x_2(t)$, respectively, where for simplicity both functions are assumed to be strictly increasing and differentiable. Suppose that the distributions of fitness values (to sex $A$) of
$B_1$ and $B_2$, respectively, are given by probabilities $P_1$ and $P_2$ that do not change with the sizes of the subpopulations, i.e., the survival and fitness functions do not change with $t$.

Further assume for simplicity that the survival functions $S_1$ and $S_2$ for $P_1$ and $P_2$ are both continuous and strictly decreasing, with identical (unique) medians $m > 0$, so $S_1(m) = S_2(m) = 0.5$. In other words, exactly half of each subpopulation $B_1$ and $B_2$ has fitness value above the median $m$ to sex $A$ at all times $t > 0$, and exactly half of each has fitness values below the median.

In this deterministic framework, the composite population of sex $B$ is growing at a rate that is proportional to the fraction $p_A$ of its members that is acceptable to the opposite sex $A$. That is, with the constant of proportionality taken to be 1,

\begin{equation}
\frac{d(x_1 + x_2)}{dt} = p_A(x_1 + x_2).
\end{equation}

Similarly, both subpopulations $B_1$ and $B_2$ are growing at rates proportional to the fractions $S_i(c^*)$ of each subpopulation that are acceptable to sex $A$ at that time, i.e.,

\begin{equation}
\frac{dx_i}{dt} = x_iS_i(c^*), \quad i = 1, 2,
\end{equation}

where $c^* = c^*(t)$ is the cutoff fitness so that the proportion $p_A$ of sex $B$ is above that fitness level at time $t$.

Suppose that $P_1$ is more variable than $P_2$. Then it can be shown (see Proposition 6.3 below) that if $p_A < 0.5$, i.e., if sex $A$ is selective, then the relative rate of growth of $B_1$ exceeds that of $B_2$, and that the proportion of sex $B$ that is from $B_1$ approaches 1 in the limit as time goes to infinity. Conversely, if $p_A > 0.5$, i.e., if sex $A$ is nonselective, then the relative rate of growth of $B_2$ exceeds that of $B_1$, and the less variable subpopulation $B_2$ dominates $B_1$ in the limit.

**Example 4.** Let the survival functions $S_1$ and $S_2$ for subpopulations $B_1$ and $B_2$ be standard symmetric double-exponential (Laplace) distributions with $S_1(x) = e^{-x}/2$ for $x \geq 0$ and $S_2(x) = e^{-2x}/2$ for $x \geq 0$ (see Figure 2). Here subpopulation $B_1$ is clearly more variable than $B_2$.

Suppose first that sex $A$ is selective and accepts only the top quarter of individuals in sex $B$, i.e., $p_A = 0.25$. Using (1) and (2), and noting that $S_2(x) = 2S_1^2(x)$ for $x \geq 0$ yields the following coupled system of ordinary differential equations:
Figure 2. Variability in LaPlace distributions. The red curve is the density of the more variable subpopulation $B_1$ in Example 4, and the blue curve is the density of the less variable subpopulation $B_2$. The vertical green line is the threshold cutoff for the opposite sex $A$ so that exactly 25% of the composite $B$ population has fitness level above that point. The vertical purple line is the value so that exactly 75% of the $B$ population has fitness level above that point. Note that the fitness values of both drop off exponentially fast from the mean in both directions.

\[
\frac{dx_1}{dt} = x_1 \left( \frac{\sqrt{x_1^2 + 2x_1x_2 + 2x_2^2} - x_1}{4x_2} \right)
\]
\[
\frac{dx_2}{dt} = \left( \frac{x_1 + x_2}{4} \right) - x_1 \left( \frac{\sqrt{x_1^2 + 2x_1x_2 + 2x_2^2} - x_1}{4x_2} \right).
\]

Figure 3 illustrates a numerical solution of (3) with the initial condition $x_1(0) = x_2(0) = 1$. Note that in this case where sex $A$ is selective, the more variable subpopulation $B_1$ eventually overwhelms the less variable $B_2$.

Suppose next that sex $A$ is nonselective and accepts the top three-quarters of individuals in sex $B$, i.e., $p_A = 0.75$. Using (1) and (2) again, and noting that $S_2(x) = 4S_1(x) - 2S_1^2(x) - 1$ for $x \leq 0$ yields the following system:
Figure 3. Selective case — Population sizes and ratio. The graphs of $x_1(t)$ and $x_2(t)$ (a) and the ratio $\frac{x_1(t)}{x_1(t) + x_2(t)}$ (b) satisfying (3).

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 \left( \frac{x_1 + 4x_2 - \sqrt{x_1^2 + 2x_1x_2 + 2x_2^2}}{4x_2} \right) \\
\frac{dx_2}{dt} &= \left( \frac{3x_1 + 3x_2}{4} \right) - x_1 \left( \frac{x_1 + 4x_2 - \sqrt{x_1^2 + 2x_1x_2 + 2x_2^2}}{4x_2} \right).
\end{align*}
\]

Figure 4 illustrates a numerical solution for this case with the same initial condition $x_1(0) = x_2(0) = 1$. Note that in this situation where sex A is nonselective, the less variable subpopulation $B_2$ dominates the more variable $B_1$.

Figure 4. Nonselective case — Population sizes and ratio. The graphs of $x_1(t)$ and $x_2(t)$ (a) and the ratio $x_1(t)/(x_1(t) + x_2(t))$ (b) satisfying (4).

Note that the birth process model above also implicitly includes simple birth-death processes, via the simple observation that a population growing, for example, at a rate of eight per cent and dying at a rate of three per cent, can be viewed as a pure birth process growing at a rate of five per cent.
6. Proofs

Probabilistic model proofs.

Proposition 6.1. \( N(\mu, \sigma_1^2) \) is more variable than \( N(\mu, \sigma_2^2) \) if and only if \( \sigma_1^2 > \sigma_2^2 \).

Proof. Let \( X_1 \sim N(\mu, \sigma_1^2) \) and \( X_2 \sim N(\mu, \sigma_2^2) \). Then for all \( c > \mu \),

\[ P(X_1 > c) = P(\sigma_2(X_1 - \mu) > \sigma_2(c - \mu)) = P(\sigma_1(X_2 - \mu) > \sigma_2(c - \mu)) > P(\sigma_1(X_2 - \mu) > \sigma_1(c - \mu)) = P(X_2 > c), \]

where the second equality follows since, by the rescaling and translation properties of normal distributions,

\( \sigma_2(X_1 - \mu) \) and \( \sigma_1(X_2 - \mu) \) are both \( N(0, \sigma_1^2 \sigma_2^2) \).

The case \( c < \mu \) follows similarly, and since the mean of every normal distribution is the same as the median, this completes the proof. \( \square \)

Suppose that \( X_1 \) and \( X_2 \) are the fitness values of two random individuals chosen from \( B_1 \) and \( B_2 \), respectively, with \( X_1 \sim N(\mu, \sigma_1^2) \) and \( X_2 \sim N(\mu, \sigma_2^2) \). (Note the key assumption that the average values, i.e., the medians, are the same.)

Proposition 6.2. Suppose subpopulation \( B_1 \) is more variable than \( B_2 \), i.e., \( \sigma_1^2 > \sigma_2^2 \). Then

(i) If \( p_A < 0.5 \), i.e., if sex \( A \) is selective, then the probability that a random individual from \( B_1 \) is selected for mating by sex \( A \) is greater than the probability that a random individual from \( B_2 \) is selected.

(ii) If \( p_A > 0.5 \), i.e., if sex \( A \) is nonselective, then the probability that a random individual from \( B_2 \) is selected for mating by sex \( A \) is greater than the probability that a random individual from \( B_1 \) is selected.

Proof. Let \( X_1 \) and \( X_2 \) denote the fitness values for an individual drawn at random from \( B_1 \) and \( B_2 \), respectively, and let \( S_1 \) and \( S_2 \) denote the survival functions for \( X_1 \) and \( X_2 \). First, it will be shown that there exists a unique “threshold” fitness cutoff \( c^* \in \mathbb{R} \) such that

\[ \beta S_1(c^*) + (1 - \beta) S_2(c^*) = p_A, \]

\( c^* > \mu \) if \( p_A < 0.5 \) and \( c^* < \mu \) if \( p_A > 0.5 \).

To see (5), let \( g : \mathbb{R} \rightarrow (0, 1) \) be given by \( g(c) = \beta S_1(c) + (1 - \beta) S_2(c) \). Then \( g \) is continuous and strictly decreasing with \( g(-\infty) = 1 \), \( g(\mu) = 0.5 \), \( g(\infty) = 0 \), so \( c^* \) satisfying (5) exists and is unique, and since

\( S_1(\mu) = S_2(\mu) = 0.5 \),

\( c^* > \mu \) if \( p_A < 0.5 \) and \( c^* < \mu \) if \( p_A > 0.5 \).
Next, note that from (5) and the definitions of $p_A$, $\beta$ and $c^*$, it follows that
\[ X_i \text{ is acceptable to sex } A \text{ if and only if } X_i > c^*, \quad i = 1, 2. \]
To see (i), assume that $p_A < 0.5$ and note that
\[
P(X_1 \text{ is acceptable to sex } A) = P(X_1 > c^*)
\]
\[
> P(X_2 > c^*) = P(X_2 \text{ is acceptable to sex } A),
\]
where the first and last equalities follow by (6), and the inequality by Proposition 6.1 since $B_1$ is more variable than $B_2$. Conclusion (ii) follows analogously. □

Deterministic model proof.

Proposition 6.3. Suppose subpopulation $B_1$ is more variable than $B_2$, i.e., $P_1 > P_2$.

(i) If $p_A < 0.5$, i.e., if sex $A$ is selective, then the relative rate of growth of $B_1$ exceeds that of $B_2$,
\[
\frac{1}{x_1} \frac{dx_1}{dt} > \frac{1}{x_2} \frac{dx_2}{dt}.
\]
Moreover, $\frac{x_1}{x_1 + x_2} \to 1$ as $t \to \infty$.

(ii) If $p_A > 0.5$, i.e., if sex $A$ is nonselective, then the relative rate of growth of $B_2$ exceeds that of $B_1$,
\[
\frac{1}{x_2} \frac{dx_2}{dt} > \frac{1}{x_1} \frac{dx_1}{dt}.
\]
Moreover, $\frac{x_1}{x_1 + x_2} \to 0$ as $t \to \infty$.

Proof. (i) Analogous to the argument for Proposition 6.2, define $g : \mathbb{R} \to (0, 1)$ by
\[
g(c) = \frac{x_1 S_1(c) + x_2 S_2(c)}{x_1 + x_2},
\]
where $S_1$ and $S_2$ are the survival functions for $P_1$ and $P_2$, respectively. Recall that $S_1$ and $S_2$ are both continuous and strictly decreasing with identical medians $m > 0$, and fix $t > 0$. Since $g$ is continuous and strictly decreasing with $g(-\infty) = 1$, $g(m) = 0.5$, and $g(\infty) = 0$, there exists a unique threshold fitness cutoff $c^* = c^*(t)$ satisfying
\[
\frac{x_1 S_1(c^*) + x_2 S_2(c^*)}{x_1 + x_2} = p_A,
\]
where, as before, $p_A$ is the upper fraction of sex $B$ that is acceptable to sex $A$, and $c^* = c^*(t)$ is the threshold fitness cutoff for sex $A$ for the combined populations of sex $B = B_1 \cup B_2$ at time $t$.

Note that $S_1(m) = S_2(m) = 0.5$, so since $p_A < 0.5$, $c^* > m$. Since $P_1 > P_2$, this implies that $S_1(c^*) > S_2(c^*)$. Since $S_1(c^*)$ and $S_2(c^*)$ are the
proportions of $B_1$ and $B_2$, respectively, that are above the threshold cutoff at time $t > 0$, (2) implies (7).

To see that $\frac{x_1}{x_1 + x_2} \to 1$ as $t \to \infty$, note that $P_1 > P_2$ implies that

\[ m < S_2^{-1}(p_A) < S_1^{-1}(p_A) \text{ for } p_A < 0.5. \]

Clearly $c^* \in [S_2^{-1}(p_A), S_1^{-1}(p_A)]$ for all $t > 0$, so since $S_2(x) < S_1(x)$ for all $x > m$, the continuity of $S_1$ and $S_2$ implies the existence of $\delta > 0$ so that

\[ S_1(c^*) > S_2(c^*) + \delta \text{ for all } c^* \in [S_2^{-1}(p_A), S_1^{-1}(p_A)] \text{ and for all } t > 0. \]

Thus by (2),

\[ \frac{1}{x_1} \frac{dx_1}{dt} > \frac{1}{x_2} \frac{dx_2}{dt} + \delta \text{ for all } t > 0, \]

so $\ln x_1 - \ln x_2 \geq \delta t + \alpha$, which implies that $\frac{x_1}{x_1 + x_2} \to 1$ as $t \to \infty$, completing the proof of (i).

The proof of (ii) is analogous. \qed

7. Differences in gender variability

The selectivity theory of differences in gender variability introduced here explains how current greater or lesser variability could depend on the past selectivity factor of the opposite sex, and as such pertains equally to any species and either sex. If both sexes in a certain species began with comparable mid-range variability, for example, and if its females were generally selective ($p_F < 0.5$) or its males were generally nonselective ($p_M > 0.5$), or both, this would have led to relatively greater male variability, i.e., the VH. Thus if there were a biological reason, such as the parenting factors described above, for either or both of such gender patterns in selectivity to have occurred over time, the above selectivity theory would predict a species whose males currently exhibit relatively more variability than its females.

Moreover, if this selectivity theory has validity, gender differences in variability are time dependent whenever the two sexes’ tendencies in selectivity are evolving. If gender differences in selectivity have been decreasing and are now less significant in some species than they were in prehistoric times, then this theory could also suggest that the gender difference in variability in those species has also been decreasing, i.e., the VH has been slowly disappearing.

8. Conclusions

The goal here has been neither to challenge nor to confirm the VH, but rather to propose an elementary mathematical theory based on biological/evolutionary mechanisms that might help explain how one gender of a species might tend to evolve with greater variability than the other gender. Bear in mind that the precise formal definitions and assumptions made here are clearly not applicable in real-life scenarios, and that the contribution here is thus also merely a general theory based on unproved and unprovable
hypotheses. This theory is independent of species, and although it may raise red flags for some when applied to homo sapiens, we share the viewpoint of Eriksson et al that the “variability hypothesis is not only of mere historical interest but also has current relevance for clinical practice” [EMTA12, p. 329].

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References


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