

# WORKING WITH LEARNERS' MATHEMATICS: EXPLORING A KEY ELEMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING

<u>Jill Adler</u>	Zain Davis	Mercy Kazima	Diane Parker	Lyn Webb
University of the Witwatersrand	University of Cape Town	University of the Witwatersrand	University of Kwazulu-Natal	Nelson Mandela Metropolitan

*This paper explores an element of mathematics for teaching (MfT), specifically 'interpreting and judging students' mathematical productions'. The research reported draws from a wider study that includes an examination of MfT produced across teacher education sites in South Africa. We show that this element of MfT is privileged across sites, evidence that it is valued in teacher education practice. Its production varies, however, enabling elaboration of this element of MfT.*

## INTRODUCTION

A distinguishing feature of mathematics teacher education is its dual, yet deeply interwoven, objects: *teaching* (i.e. learning to *teach* mathematics) and *mathematics* (i.e. learning *mathematics* for teaching (*MfT*)) – the subject-method tension. These dual objects, and their inter-relation are writ large in *in-service* teacher education (INSET) programs where new and/or different ways of knowing and doing school mathematics combine with new and/or different contexts for teaching. Such are the conditions of continuing professional development in South Africa. Post apartheid South Africa has seen a proliferation of formal (i.e., linked to accreditation) and informal INSET programs. Debate continues as to whether and how these programs should integrate or separate out opportunities for teachers to (re)learn *mathematics* and *teaching*. Programs range across this spectrum, varying in degree to which opportunities for teachers to learn are embedded in problems of (mathematics teaching) practice, and so opportunities for learning more of their specialized knowledge, MfT.

In the QUANTUM research project, we are currently studying mathematics and mathematics education courses in three mathematics teacher education sites where the programs differ in relation to their integration of mathematics and teaching. The goal is not to measure impact of these different approaches, but rather, through in-depth investigation of practices within these courses, to understand *what* and *how* *mathematics* and *teaching* come to be (co)produced across and within these settings. We are thus examining practices *inside* teacher education. Specifically, and this is discussed further below, we are investigating how and what knowledge(s) are appealed to as elements of MfT come to be legitimated in pedagogic discourse.

Our focus in this paper is on one privileged MfT practice evident across all three sites: working with learners' mathematical productions. Learners' mathematical productions, and teachers' engagement with these, have been a prevalent theme in mathematics education research and widely reported in PME. In this paper we assume the importance of teachers being able to do this work. The concern, rather, is with the mathematical entailments of this work and its elaboration in teacher education practice. Our examination of the practices across three sites reveals that while the notion of working with (interpreting, analyzing, judging) student mathematical thinking is common, it emerges and is approached in quite different ways, illuminating this element of MfT in interesting ways. Our observations are a function of a particular analytic tool, and its underlying theoretical orientation both of which are elaborated below. We begin with a brief discussion of QUANTUM – the wider research project.

## THE QUANTUM PROJECT

The overarching 'problem' under scrutiny in QUANTUM<sup>1</sup> is *mathematics for teaching (MfT)*, its principled description and related opportunities for teachers' learning. We regard the mathematical work of teaching as a particular kind of mathematical problem-solving<sup>2</sup> - a situated knowledge, shaping and being shaped by the practice of teaching. More specifically we are concerned with the mathematics middle and senior school teachers need to know and know how to use (i.e. the mathematical work they do) in order to teach mathematics well in diverse classroom contexts in South Africa; and with how, and in what ways, programs that prepare and/or support mathematics teachers provide opportunities for learning MfT. Elsewhere (Adler, Davis & Kazima, 2005), we have problematised the renewed focus on subject knowledge for teaching in mathematics education, its development from Shulman's seminal work on pedagogic content knowledge, how it remains underdescribed, and how mathematics teacher education practice, as well as school teaching practice, is a productive empirical site in the project.

In our earlier work (Adler & Davis, 2004) we exemplified a pedagogic practice where learners are expected to engage with novel mathematics problems, and showed that meanings can and do proliferate. The teacher has considerable mathematical work to do as s/he navigates between varying learner responses, and what would constitute a robust mathematical solution. S/he needs to figure out how to mediate between these interpretations, and the mathematical notion(s) and dispositions she would like all learners in the class to consolidate. S/he needs to figure out suitable questions to ask learners, or comments to make. Both have mathematical entailments.

Ball, Bass and Hill (2004, p.59) describe these mathematical practices as elements of the specialised mathematical problems teachers solve as they teach. These elements

---

<sup>1</sup> For more detail on QUANTUM see Adler & Davis (2004)

<sup>2</sup> We thank Deborah Ball for this description – personal communication, Adler and Ball.

include the ability to “design mathematically accurate explanations that are comprehensible and useful for students” and “interpret and make mathematical and pedagogical judgements about students’ questions, solutions, problems, and insights (both predictable and unusual)”. They posit a more general feature, “unpacking”, as an essential and distinctive feature of “knowing mathematics for teaching”.<sup>3</sup> We have already noted the extensive work in the field of mathematics education on learners’ constructions of mathematical ideas and related work on misconceptions (e.g. Smith, DiSessa, & Roschell, 1993). There has been far less attention, in our view, to the kinds of mathematical and pedagogical judgements teachers make as they go about their work on student productions<sup>4</sup>, hence our methodology and focus.

Our overarching theoretical orientation is elaborated in Davis, Adler, Long & Parker (2003) and Adler & Davis (2004). Briefly, the tool emerges from our use of Basil Bernstein’s sociological theory of pedagogy. We recruit Bernstein’s (1996) proposition that the whole of the pedagogic device (distribution of knowledge; rules for the transformation of knowledge into pedagogic communication) is condensed in evaluation. In other words, any pedagogy transmits evaluation rules. Additionally, evaluation is activated by the operation of pedagogic judgement by both teacher and student.

In QUANTUM we are looking at evaluative events across teacher education programs, on the assumption that these would reveal the kind of mathematical and teaching knowledge that comes to be privileged. Figure 1 presents a network of part of the tool<sup>5</sup> we are using, and includes the codings we refer to in the next section. We have highlight categories of the network pertinent to our focus in this paper. The network reflects our dual and simultaneous focus on mathematics and teaching as specialised activities, and how they emerge as objects of study over time in each of the courses. Each course, all its contact sessions and related materials, were analysed, and chunked into what we have called *evaluative events*. These are marked by punctuations in pedagogic discourse, when meanings are set through pedagogic judgement. Space limitations prevent description of the full network, and the systematic chunking done.

---

<sup>3</sup> In Adler & Davis (2004) we report QUANTUM: Phase 1. We focused on formal assessment tasks across math and math education courses in 11 institutions in South Africa. A key ‘finding’ is that across courses, formal assessments of unpacked mathematics in relation to teaching were very limited.

<sup>4</sup> A very recent study by Karin Brodie (Brodie 2005) has explored teacher moves as they engage learner thinking. Her analysis provides an important description of this work of mathematics teaching.

<sup>5</sup> Missing here is an additional set of columns on *subject positions*. These are significant in their relation to particular notions and how they unfold over time, and are the focus of a different paper. See Adler, Davis, Kazima, Parker & Webb, forthcoming.

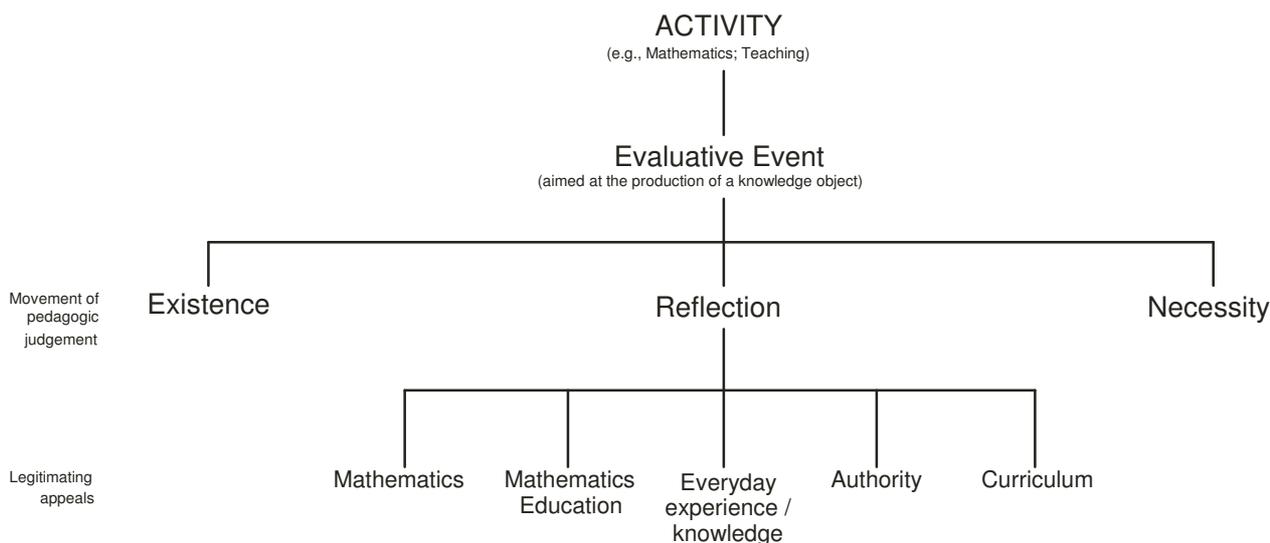


Figure 1: Network describing the movement of pedagogic judgement

Suffice it to say that for each event, we coded first whether the object was a mathematical (M) and/or teaching (T) one, or both, and then whether elements of the object(s) were assumed known, rather than being focus of study (and were then coded either m or t). The additional branches in the network emerge through a recontextualisation of Hegel’s theory of judgement (1969). We recruit from Hegel the proposition that judgement in general, and hence pedagogic judgement in particular, is itself constituted by a series of dialectically entailed judgements (of Existence, Reflection, Necessity, and the Notion). Here we are working with the idea that in pedagogic practice, in order for something to be learned, to become known, it has to be represented. Initial orientation to the object, then, is one of immediacy – it exists in some initial (re)presented form, and can only be grasped as brute Existence. Pedagogic interaction (Reflection) then produces a field of possibilities for the object, and through related judgements made on what is and is not the object (Legitimizing Appeals), so possibilities are generated (or not) for learners to grasp the object (Necessity).<sup>6</sup> In other words, the legitimating appeals can be thought of as qualifying reflection. An examination of what is appealed to and how appeals are made in the teaching of mathematics delivers up insights into how MfT is being constituted in teacher education.

## WORK ON LEARNER MATHS ACROSS THREE COURSES

Table 1, p.8 provides summary information about the course on each site. The last three rows provide a description of the analysis of our data set, particularly in relation to where and how legitimating appeals are made. Each course is for in-service teachers, and part of a larger program towards a qualification. Two courses are aimed at Senior Secondary teachers, one at junior secondary; two are level 6

<sup>6</sup> All judgement, hence all evaluation, necessarily appeals to some or other locus of legitimation to ground itself, even if only implicitly.

(undergraduate) and 1 level 7 (post graduate) courses. They share similar goals (to provide learning experiences that will enable and improve mathematics teaching), with the Level 7 course having an additional academically oriented goal.

The Algebra concepts and methods course (Site 1) is concerned with algebraic thinking at the Grades 7 – 9 level. The underlying assumption in this course, guided by the teachers being primary trained, is that the teachers were unlikely to be adept in algebraic thinking, though they would, like their learners have learned algebraic rules as recipes. They thus needed to learn this way of thinking mathematically. They also needed to learn how to teach this in Grade 7 – 9 classes. These dual goals were integrated in a pedagogic practice that provides experiences for teaching/learning algebra that model the pedagogic practice teachers could/should use in their own classrooms. Teachers could then learn the mathematics needed and at the same time experience how it should be taught. In each of the course sessions dealing with patterns, teachers were given three or four possible formulae that could be generated from a given sequence as if these were produced by learners. Teachers were asked to visualize and explain how each different learner was thinking. In sessions dealing with algebraic rules and operations, teachers were informed of typical learner errors (explained as a result of learning ‘recipes’), and provided a way of dealing with these errors. For example, in order to clarify and prevent wrong application of laws of indices, learners could be shown how and why the rule worked (i.e. test it) through substitution of appropriately selected (small) numbers. As indicated in Table 1, legitimating appeals are made to mathematics and everyday life. It is interesting, firstly, that there are moments where everyday experience is appealed to for legitimating mathematical knowledge (specifically algebraic thinking); and secondly when the appeal is mathematical, it is restricted to numerical examples appropriate to learners at Grades 7 – 9.

In Site 2, The Professional Practice in Mathematics Education course provides a structured guide to an action research project teachers are to do. One element of the structured guide is what is referred to as a hypothetical learning trajectory (HLT) – a global teaching practice that includes ways of eliciting student knowledge, generating possible student responses, and analysing student work. As preparation for the weekend session where this aspect of their research was in focus, teachers were meant to bring examples from their own practice where they had elicited student thinking and analysed it. The course materials carried reading on misperceptions. Few teachers brought their preparation<sup>7</sup>. With only some having these available for reflection in the session, the lecturer produced an example of an HLT based on decimal fractions so that all teachers had some object to reflect on. In other words, she provided a model or demonstration of an HLT and related learner productions. Hence the coding of the legitimating appeals in relevant events in this course being described in Table 1 as either teachers’ own experience, or a demonstration/assertion

---

<sup>7</sup> In all three courses, there were sessions where lecturers commented on the importance of the teachers doing the preparation work required.

by the lecturer (authority). The interesting issue here is that the practice that emerges is a function of both the assumptions in the course, and how the teachers respond to demands on them<sup>8</sup>.

In the course in Site 3 on Mathematical Reasoning, there were 9 events, over three sessions, with one session entirely on misconceptions. Teachers' experience is the initial resource called in in the introductory session – they were given a task (A learner says that  $x^2 + 1$  cannot be zero if  $x$  is a real number. Is s/he correct?) and asked to reflect on the kinds of misconceptions their learners were likely to make as they did the task. They were also required to read Smith et al's paper on misconceptions. These together begin to generate a wide field of possible meanings. As the session progresses, the notion of misconceptions is evaluated by appeals to research in mathematics education (classification of misconception types, empirical and theoretical arguments), mathematics itself (complex numbers, justification as testing single cases, justification as generalized argument), curriculum levels (at which complex numbers can be engaged), and records of teaching (a videotape of another teacher working with the same task). It is important to remember that this course is a graduate course. Teachers are thus expected to engage teaching and mathematics (indeed are apprenticed into) discursively. It is nevertheless interesting that it is in this course too where advanced mathematical work is drawn on in the production of MfT in relation to school learners' work.

## DISCUSSION

As a study set up to explore the (co)production of mathematics and teaching, we expected legitimating appeals to shift between these two domains. We were surprised, however, at the spread of appeal domains both in relation to mathematics, and to teaching. Across the three courses, appeals included mathematics as would be expected. We were interested to see how this was constrained in pedagogical practice when teaching was being modelled. Mathematics here was then restricted to the levels at which learners would be learning. And there were expected appeals to mathematics education as a disciplinary field, though in effect, in only one of the courses. Ideas about misconceptions in the other two remained at the level of examples provided in the course notes or by the lecturer, and recognized by teachers from their own experience. It is also of interest, that in relation to learners' thinking, there was only one instance of an appeal to curriculum knowledge. This was in Site 3 where learners' responses to the task were considered relative to curriculum levels.

As emphasized at the beginning of this paper, our concern here is neither to compare nor judge of the mathematical and teaching practices in these three courses. It is rather to understand how and why they work as they do. Space limitations prohibit

---

<sup>8</sup> We note here that, as the course progresses, the lecturer is increasingly aware of the difficulties in the approach, and adaptations needed for the teachers to progress with their action research.

further discussion here. In the presentation of this work, we will reflect further on the questions that arise from our progress so far.

## References

- Adler, J. & Davis, Z. (2003). An analysis of the structuring of evaluative tasks: A focus on level 6 mathematics INSET. In Jaffer, S. and Burgess, L. (Eds.) *Proceedings of the Ninth National Congress of the Association for Mathematics Education of South Africa*. Cape Town: AMESA
- Adler, J. & Davis, Z. (2004). Opening another black box: Researching mathematics for teaching in mathematics teacher education. Presentation to AERA panel on teacher education. San Diego. April 2004.
- Adler, J., Davis, Z., & Kazima, M. (2005). The re-emergence of subject knowledge for teaching, its significance and an agenda for research. Working paper #6. QUANTUM. University of the Witwatersrand. Johannesburg.
- Adler, J., Davis, Z., Kazima, M., Parker, D., & Webb, L. (in press). Competence and performance in mathematics teacher education pedagogies: Implications for equity. Paper submitted to MES Conference. 2005.
- Ball, D., Bass, H. & Hill, H. (2004). Knowing and using mathematical knowledge in teaching: Learning what matters. In Buffler, A. and Lausch, R. (Eds.) *Proceedings for the 12<sup>th</sup> Annual Conference of the South African Association for Research in Mathematics, Science and Technology Education*. Durban. SAARMSTE.
- Bernstein, B. (1996). *Pedagogy, Symbolic Control and Identity: Theory, Research, Critique*, London: Taylor & Francis
- Brodie, K. (2005). Learner contributions in mathematics classrooms. In *Proceedings of the 13<sup>th</sup> SAARMSTE Conference*. Unesco Windhoek Office. Windhoek.
- Davis, Z., Adler, J., Parker, D. & Long, C. (August, 2003). Elements of the language of description for the production of data. QUANTUM Research Project. Working paper #2. Johannesburg: University of the Witwatersrand.
- Hegel, GWF (1969). *Hegel's Science of Logic*, translated by A.V. Miller, New York: Humanity Books
- Smith, J.P., DiSessa, A.A. & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The journal of the learning sciences*, 3,2, 115-163.

Table 1	Site 1	Site 2	Site 3
Course topic	Algebra concepts and methods	Professional practice in mathematics education	Teaching and learning mathematical reasoning
Qualification	Level 6: ACE Advanced Certificate in Education	Level 6: ACE	Level 7: Hons degree in Math Education
Time and texts	7 X 2 hr contact sessions; Course booklet	Distance learning: written materials; bi-weekly w/end sessions; 10 weeks	7 X 3 hr contact sessions; course reader
# students	25	25	30
Comments on teachers	Experienced elementary teachers upgrading initial diploma to degree level, with qualifications to teach through Grade 9	Experienced secondary teachers upgrading from initial 3 year diploma, to degree equivalent qualification.	Experienced secondary teachers extending 4 year qualification to Honours–first level graduate study
Integration of M and T	Mathematics and teach integrated within a course.	Math and math ed courses separated, with maths courses taught in the Maths Department	M and T courses Separated, each with strong ‘eye’ on other. Most taught by maths ed staff. Geom taught by tertiary math lecturer
Assumptions and relation to practice	Algebra is focus of course. Algebra is taught to teachers as they would be expected to teach it to Grade 7 – 9 learners. Embedding in practice is thus through <i>modelling the practice</i> .	Improving knowledge and practice through systematic <i>reflection on own teaching</i> experience. Embedding in practice is hypothetical, assuming teachers can generate problems and related records of practice	Mathematics teaching treated as a discursive practice, that can and should be studied. Embedding in practice is studying research in the field, and records of practice generated from outside of teachers themselves.
Events, appeals and mathematical entailments	10 events identified; appeals mainly to math, restricted, however, to the level of learners. MfT algebra restricted to testing rules with appropriate numerical examples, and so a level of mathematical work that remains at the level of the learners.	4 events where appeals are to teachers’ own experience at the start, and in the end to the lecturer modelling /demonstrating a particular instance (generated by the lecturer) of an HLT and related learner work.	9 events where appeals are to Mathematics itself, including advanced mathematics (complex numbers) and justifications; to curriculum (what learners are expected to know at what levels); to research in mathematics education; as well as initially to teachers own experience.