

A DEVELOPMENTAL MODEL FOR PROPORTIONAL REASONING IN RATIO COMPARISON TASKS

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The main purpose of this paper is to propose a model that could describe the mode in which people acquire the ability for proportional reasoning. The framework and the analysed data are part of an ongoing research, in which the responses of subjects of different ages and schoolings to different ratio- and rate-comparison tasks are studied. A special accent is placed on the influence of number structure and context upon proportional reasoning; the proposed model is based on a classification of number structure reported in PME-26 and on a classification of contexts in three categories (Rate, Mixture and Probability problems).

This paper reports part of an ongoing research on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks. In the part conveyed, we are concerned with the following question: Is it possible to describe the way in which the ability for proportional reasoning develops? The results reported and the ensuing proposed model are part of a larger study (Alatorre, 2004).

FRAMEWORK, PROVIDED BY PREVIOUS WORK

In the last three PME's different parts of the research have been put forward. The framework used in the research was presented in Alatorre (2002), an explanation of what are the "different kinds of ratio comparison tasks" as well as a description of the interview protocol used in the experimental part were submitted in Alatorre and Figueras (2003), and in Alatorre and Figueras (2004) the results obtained by six quasi-illiterate adult subjects were described. A succinct summary of these papers will be sketched here; the reader is referred to them for a more complete account.

Among the problems calling for proportional reasoning, those in which the task is a comparison of ratios can be classified according to three issues: context, quantity type, and numerical structure. Figure 1 proposes a classification according to the first one; it blends together the classifications proposed by several authors (Freudenthal, 1983; Tourniaire and Pulos, 1985; Lesh, Post and Behr, 1988; Lamon, 1993).

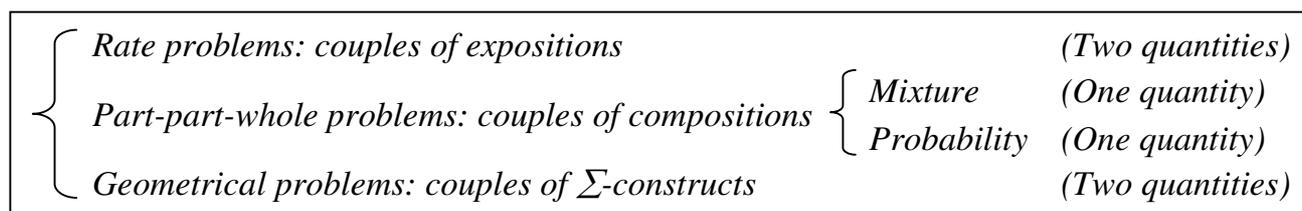


Figure 1: Taxonomy of ratio comparison tasks according to context

Examples of the first three kinds are shown in Figures 2 to 4 (geometrical problems are not dealt with in this research).

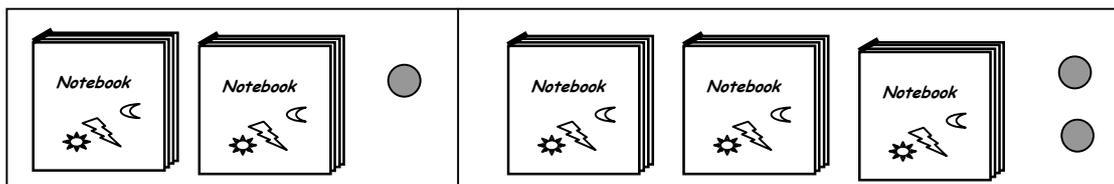


Figure 2. Example of a Rate problem: In which store are the notebooks cheaper? (The round figures stand for coins)



Figure 3. Example of a Mixture problem: In which jar does the mixture taste stronger? (The grey glasses contain concentrate and the clear ones contain water). (Problem taken from Noelting, 1980)

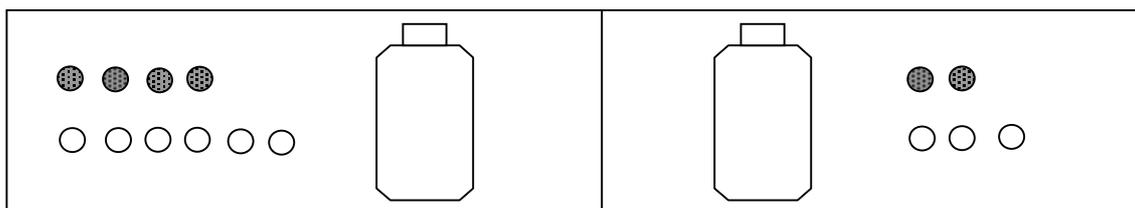


Figure 4. Example of a Probability problem: If bottles are shaken with marbles inside, in which one is a dark marble more likely to come out at the first try?

The second issue is the quantity type. Quantities can be discrete (as the marbles in Figure 4) or continuous (as the amounts of liquids in Figure 3).

The third issue is the numerical structure. In a ratio or rate comparison there is always a foursome: four numbers stemming from two “objects” (1 and 2), in each of which there is an antecedent (e.g. notebooks, concentrate glasses, dark marbles) and a consequent (e.g. coins, water glasses, light marbles). Alatorre’s (2002) framework includes a classification of all possible such foursomes in 86 different *situations* that can be grouped in three difficulty levels, labelled L1, L2, and L3; their description will close the section dedicated to the framework.

In the previous paragraphs a description of the classification of ratio-comparison problems was given. Here follows a classification of the strategies used by subjects in their answers to such problems. Alatorre’s (2002) framework, as presented in Alatorre and Figueras (2003 and 2004), is to be used. Strategies can be simple or composed; in turn, simple strategies can be centrations or relations. Centrations can be on the totals CT, on the antecedents CA, or on the consequents CC. Relations can

be order relations RO (when an order relationship is established among the antecedent and the consequent of each object and the results are compared), or subtractive relations RS (additive strategies), or proportionality relations RP. Composed strategies can be four forms of logical juxtapositions of two strategies.

Strategies may be labelled as correct or incorrect, sometimes depending on the situation (combination and location) in which they are used. The most important correct strategies are:

- RP in all situations (for instance, saying in Figure 2 that in side 1 the notebooks are cheaper because they cost \$0.50, whereas in side 2 they cost \$0.67; or saying in Figure 3 that side 2 has a stronger taste because if three times as much juice was prepared in jar 1 it would need the same three concentrate glasses that are in jar 2, but twelve water glasses, which are more than the two of jar 2; or saying in Figure 4 that in both bottles a dark marble is equally likely, because side 1 is twice as much as side 2, or because in both sides there are three light marbles for every pair of dark ones);
- RO in situations where one of the antecedents equals its consequent, or where one of the antecedents is less than its consequent and the other is more than its consequent (for instance, in Figure 3, saying that jar 2 has a stronger taste because it has more concentrate than water, whereas jar 1 has more water than concentrate);
- In some situations, some composed strategies that can be considered as theorems in action (TA, see e.g., Vergnaud, 1981) (for instance, saying in Figure 3 that jar 2 has a stronger taste because it has more concentrate *and* fewer water glasses than jar 1); there are overall 14 TA's.

Incorrect strategies are:

- CT in all situations (for instance, saying in Figure 4 that a dark marble is more likely in bottle 1 because it has altogether more marbles than bottle 2);
- CA in most situations (for instance, saying in Figure 2 that side 2 is cheaper than side 1 because it has more notebooks than side 1);
- CC in most situations (for instance, saying in Figure 4 that a dark marble is more likely in bottle 2 because it has fewer light marbles than bottle 1);
- RO in most situations (for instance, saying in Figure 2 that in both sides the notebooks are equally cheap because both have more notebooks than coins);
- RS in all situations (for instance, saying in Figure 4 that a dark marble is more likely in bottle 2 because it only has one more light marble than dark ones, whereas in side 1 there are two more);
- Most composed strategies.

The three difficulty levels mentioned before refer to which correct strategies may be applied. L1 consists of all the situations where, in addition to RP, other correct strategies may be used. In L2 and L3 only RP can be used; the difference among

them is that L2 consists of situations of proportionality (both ratios or rates are the same), and L3 consists of situations of non-proportionality. An example of L1 is the array of Figure 3; an example of L2 is Figure 4; and an example of L3 is Figure 2.

METHODOLOGY

A case study was conducted in Mexico City with 23 subjects, aged from 9 to 65 and with schooling from 0 (illiterate adults) to 23 (PhD). Each one was interviewed for a time between 60 and 90 minutes; the sessions were videotaped. Two of the subjects are in fact one, Sofía, who was interviewed twice: when she was aged 10 and 12.

During the interviews, subjects were posed several questions in each of 10 sorts of problems, which were 4 Rate problems (of which the juice problem of Figure 2), 2 Mixture problems (of which the notebook problem of Figure 3), 2 Probability problems (of which the marbles problem of Figure 4), and two forms of partitions problems as controls (one fraction and one pizza problem).

Each of the ten problems was posed in different questions according to numerical structure. Fifteen such questions were designed, five in each of the difficulty levels L1, L2, and L3; all the problems could be posed in each of them. To each subject all of the problems were posed in some of the 15 numerical questions, covering at least a couple of the questions of each level. Each time, the subjects were asked to make a decision (side 1, side 2, or “it is the same”) and to justify it.

A total of 2518 answers was thus obtained; 2049 (81%) of them were classified using the strategies system described above, and the rest either consisted of a decision without a justification (9%), or had a justification that was only a description (4%), or consisted of solution mechanisms different from the strategies described before (6%). Two phases of analysis were undertaken: quantitative and qualitative.

QUANTITATIVE ANALYSIS

In order to make a quantitative analysis possible, one point was given to all correct strategies, and 0.5 point was given to answers that could be incomplete expressions of correct theorems in action. Also, 0.5 point was given to all non-classifiable answers that fulfilled the following conditions: correct decision and either no mechanism or a mechanism that could eventually become correct (such as arithmetic or geometric approximations). Then, for each group of answers (e.g., for each subject) a score was obtained, and expressed as a percentage of the answers in that group.

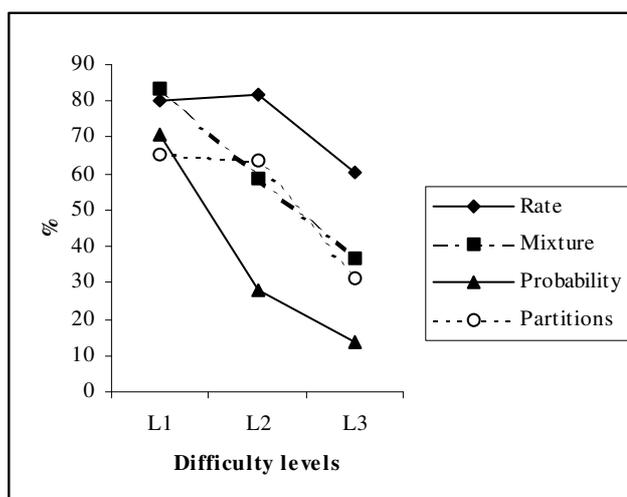


Figure 5. Scores according to context

A first approach consists of verifying that the categories labelled L1, L2, and L3 are indeed difficulty levels. As Figure 5 shows, L3 is, in all of the context types, the most difficult (i.e., the one with lowest scores), and L1 the easiest (highest scores). Except for the Rate and the Partitions (control) problems, L2 has intermediate scores. Figure 5 also allows a comparison of the different context types. The Rate problems are the easiest ones, and the Probability problems are the most difficult ones. Mixture problems are as easy as Rate problems only in level L1, and in levels L2 and L3 lay between Rate and Probability problems.

In a second approach the behaviour of the 23 subjects in the three levels (across all contexts) is studied. The 23 subjects can be classified in four groups, as shown in Figure 6, where the age (child = younger than 15, adult = older than 15) and the schooling of the subjects within each group are also described.

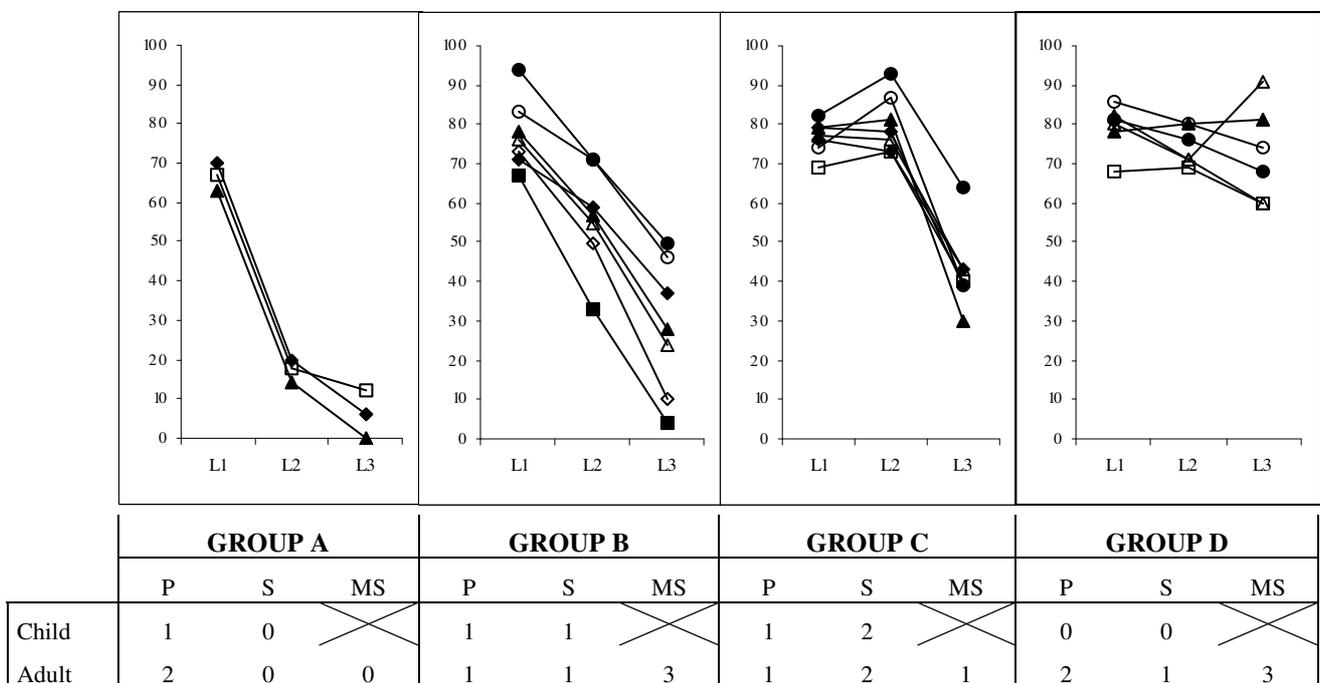


Figure 6. Four groups of subjects

(P=Primary school or less, S=Secondary school, MS=More than Secondary School)

Group A consists of three subjects in primary school, a child and two adults. For them Level L1 was fairly easy, level L3 was very difficult, and level L2 was almost as difficult as L3. Groups B and C consist of assortments of young and adult subjects of all schooling levels; they all find level L1 rather easy and level L3 rather difficult; the difference between Group B and Group C is that in the former the difficulty of level L2 lies midway between those of L1 and L3, whereas in the latter it is equal or even smaller than that of L1. Finally, group D consists of six adult subjects in the three schooling stages, who had good results in all three levels L1, L2, and L3. Among these subjects a necessary (but not sufficient) condition for belonging to Group A was very little schooling, and a necessary (but not sufficient) condition for belonging to Group D was some age (the youngest of these subjects was aged 16).

An interesting case is that of Sofía, interviewed at ages 10 and 12, because a small longitudinal study can be carried out with her data. Figure 7 shows that when aged 10 Sofía belonged to Group B, and her development over two years took her to Group C. She had an increase in her scores in all three levels, which was small for L1 and much greater in L2 and L3.

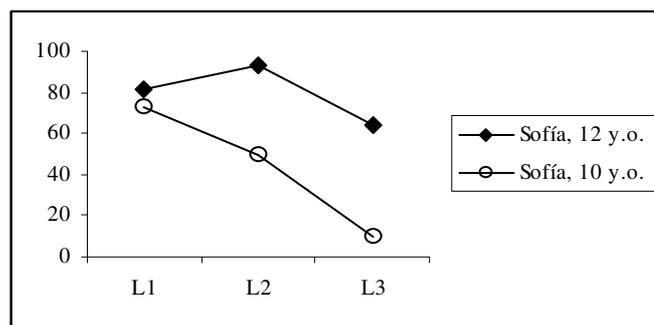


Figure 7. Sofía's results

QUALITATIVE ANALYSIS

The strategies used by the subjects of the four Groups described before differ in some ways.

- In Group A subjects use the correct order relations RO and Theorems in Action (TA) that are applicable in L1, but almost never use the proportionality relations RP. This explains their failure at levels L2 and L3, where they use mainly incorrect centrations.
- Subjects in Group B also use correct RO and TA in L1. They use RP almost only in the proportionality situations of L2, and then again only in some cases of L2 (mainly in Rate problems); they seldom use RP in the non-proportionality situations of L3. The strategies that account for the incorrect answers are mainly centrations.
- In Group C subjects use widely RP in L2, and they even use RP in some L1 questions (although still using correct RO and TA). They still use mostly incorrect strategies in L3, mainly centrations and the additive strategies RS, especially in the most difficult Probability and Mixture problems.
- Subjects of Group D can use RP in all kinds of situations. Some of the subjects go so far as to use exclusively RP, even in L1. The scarce incorrect answers are due to centrations, RS and arithmetically mistaken attempts at RP.

A DEVELOPMENTAL MODEL

The quantitative and qualitative analyses conducted permit the construction of a model that describes how the subjects grow in their ability to respond correctly to ratio comparison tasks. Subjects in groups A, B, C, and D (in that order) have increasingly higher global scores (respectively 40%, 60%, 68%, and 76%); they also use increasingly correct and sophisticated strategies. If one adds the fact that Sofía evolved from Group B to Group C, it can be postulated that within a given context, these groups correspond to stages or moments that occur in that order, as shown in Figure 8.

It may be hypothesized that before responding like Group A, which is the first moment, subjects could go through a moment zero where all three levels are equally difficult (i.e., very young children). Then the first ability to develop is the use of non-proportionality correct strategies RO and TA, which are only useful in level L1 (first moment). After that the ability to use the proportionality relations RP in proportionality situations (L2) would slowly grow, first almost without any change in the non-proportionality situations (second moment), and only when the ability to use RP in L2 equals the ability to use RP or RO or TA in L1 would the ability to use proportionality relations in the non-proportionality situations L3 start to develop (third moment). In the last stage this last ability equals that of the other two levels (fourth moment).

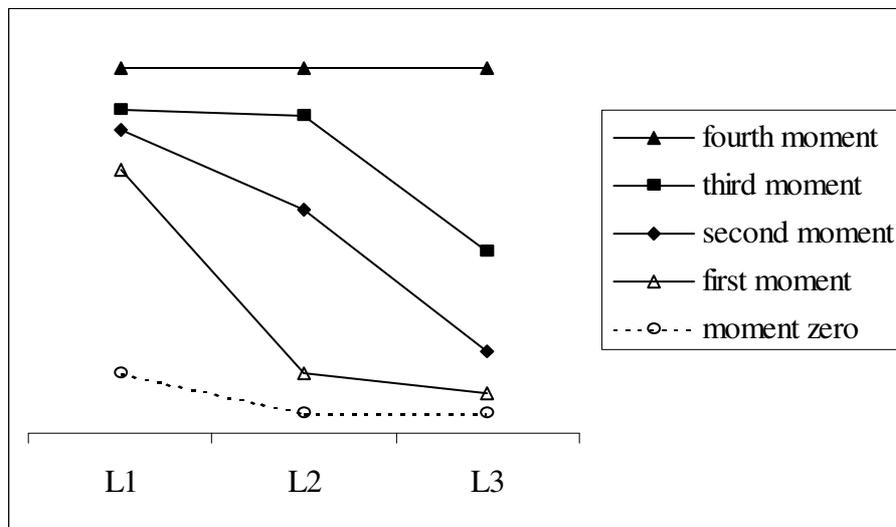


Figure 8. Developmental model

This development, however, is only within a certain kind of context. The whole process would start first with the Rate problems, which are the easiest ones, then with the Mixture problems and finally with the Probability problems, which are the most difficult ones. Thus, at a given instant a person is in different stages or moments regarding his/her response to different kinds of problems. For instance Flor, who is one of the subjects in Group D, is in the first moment in the Probability problems, in the third moment in the Mixture problems and in the fourth moment in the Rate problems (see Figure 9).

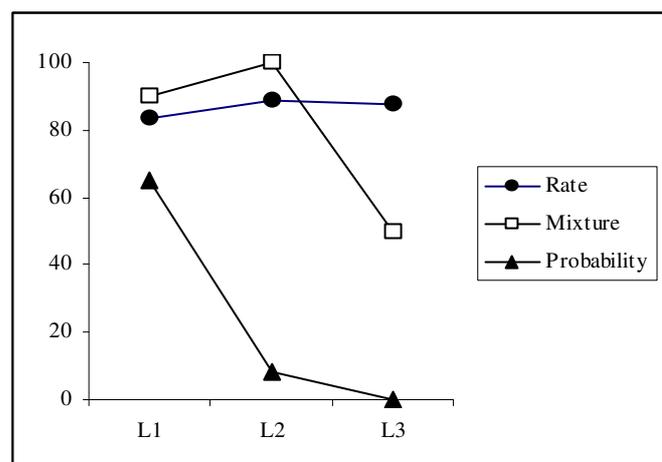


Figure 9. Flor's results

CONCLUSIONS

It has been said before that it is easier for children to solve ratio comparison tasks in proportionality situations than in non-proportionality ones. However, if one considers that other strategies apart from the proportionality relations can be correct, some non-proportionality situations can be easier (for children as well as for adult subjects) than the proportionality ones. But it is the ability to adequately solve the proportionality situations that can trigger the ability to solve the non-proportionality situations where only the proportionality strategies may be applied.

It has also been said that proportional reasoning is highly context-dependent. This paper has shown that Rate problems are the easiest to solve and Probability problems are the hardest, with Mixture problems between them.

The data obtained from this group of 23 subjects suggest that their ability for proportional reasoning evolves in the form described by the proposed developmental model. It can be conjectured that the model could describe this evolution for other subjects as well. This would in particular entail that only people with very little schooling would be at the first stage of this development, and that only people above a certain age would be at the last stage. In turn, this could imply that neither the school nor life are sufficient conditions for the development of proportional reasoning, but that both can be considered as catalysts for the process.

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