

INTERACTION OF MODALITIES IN CABRI: A CASE STUDY

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In this paper we consider an experiment in which 15 years old students explore a phenomenon of covariance, by using Cabri for drawing geometric figures, measuring, and sketching graphs. In this way they collect different types of information. A first research question is how students deal with them in situations of exploration, in particular, which role they ascribe to the numerical data. Another research question is how students approach formal aspects. The activity on which the experiment is based is a telling example of how Cabri enlarges the scope of the exploration, in a way that could not be possible in paper & pencil environment.

INTRODUCTION AND BACKGROUND

It is widely recognized that the manipulation of figures in Cabri enhances the ability of producing conjectures, see (Arzarello, 2002, De Villiers, 1997, Mariotti, 2001). But this is not all: Cabri gives access to measurement of lengths, areas etc., provides numerical data which evolve in a dynamic way, according to the changing of the corresponding geometric figure, and sketches analytical representations (diagrams) of the geometric situation. These aspects enlarge the scope of action of Cabri, since they give students the chance of studying a mathematical phenomenon from different points of view and enhance the production of conjectures in different domains. The other side of the coin is that Bruner’s three modes of representation (1966) - enactive, iconic, symbolic - are available *at the same time*. Thus the students need to manage different types of information and to transfer the information from one mode to another. This situation may also imply to switch back and forth from an operational to a structural conception, see (Sfard, 1991). The following research questions arise:

- In which way students relate geometric figures to Cartesian diagrams, and to measures?
- Which is the role ascribed to numerical data, e.g. students consider measures as an instrument for exploration of geometric situations?
- Are there moments in which the students feel the need of resorting to formulas and formalisation and, if any, which are the stimuli and the circumstances?

The mentioned studies will be the background of our attempt to answer our research questions. We will also use the representational worlds, which are related to Bruner’s modes, but seem to better apply to mathematical learning, as discussed in (Tall, 2003): embodied (related to Bruner’s enactive and iconic mode), symbolic-proceptual

and formal-axiomatic (both related to Bruner's symbolic mode). Furthermore we will take into account that Tall et al. (2001) described the path from the environment to the axiomatic mathematics as happening through two ways of operating (perception and action), which converge to reflection, and afterwards to theory. Answering our research questions means reflecting on how this path may actually develop in Cabri environment.

THE SETTING OF OUR EXPERIMENT

The students involved in the experiment are 15 years old; they attend a secondary school with a strong scientific orientation. Thanks to the innovative spirit of their teacher, they are 'frequent users' of the mathematics laboratory. Here the mathematics laboratory has to be intended not simply as a physical place, but as a 'place for learning', in which mathematical activity is carried out by different modalities (computer and other artefacts, work in groups, communication). The aim is to give students the *chance* and the *time* to construct the meaning of mathematical objects evolving from the personal meaning to the institutional meaning. In contrast with traditional mathematics teaching style, which evokes 'Fast food for mind', 'Slow food for mind' could be the motto of such a laboratory.

The students involved in the experiment have a good experience in using Cabri, working in group and exploring open situations. When using Cabri, they are accustomed to use the geometric and the analytic representations. At the moment of our experiment, they already had a first approach to calculus; in particular, they were introduced to functions, namely they knew the representation of linear and quadratic functions. During the experiment the students worked in small groups (three students for each group), each group working at one computer. The teacher sometimes intervened in the groups' activities with prompts and suggestions.

We focus on the activity of a group of three students (Nicolas, Alessio, Andrea), who, according to the teacher, are "enough good to work on the problem, but not so brilliant to solve it in a couple of minutes". They were allowed to write notes, which were collected. They were video-taped during the activity. An external observer took notes and interacted with them.

Our analysis is based on the video, the protocols, the field notes taken by the observer.

The exercise

The following statement was given to the students:

In a circle a chord AB is given. Take a point C on the circle. What may we say about the area of the triangle ABC ? Is it possible to have information on how the area of the triangle varies?

The students were accustomed to these kind of exercises and knew that the expected answer to the first question is of the type "It varies", while the answer to the second is

the description of how it varies. The Cabri figure was already on the screen of their computer (see Fig.1).

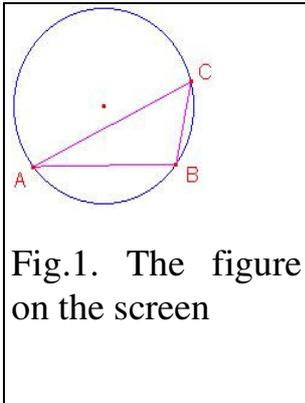


Fig.1. The figure on the screen

First of all, we observe that the exercise encompasses a situation of covariance: the area varies according to the position of the point C on the circle. Dragging with Cabri shows that the area varies according to the position of C (answer to the first question). The variation of the area may be studied with the help of a diagram done with Cabri. This implies the choice of variables. When working in Cabri it is natural to use measures (of segments, areas, angles, ...): we will see that the students chose to study the relationship involving side AC (or BC) and the area, and the relationship

between the height and the area. These choices encompass different kinds of dependence and, consequently, different diagrams (see below the discussion of the students' production): for example, the relation between the height and the area is functional; the relation between sides and area is not.

In the following, we report our microanalysis of the three students' behaviour when facing this problem. Their whole process of exploration is split into its main steps, to which we refer as 'episodes'. These episodes are presented together with our comments.

EPISODES FROM THE STUDENTS' EXPLORATION

First episode: getting on touch with the problem

The first approach to the problem is the usual one with Cabri, that is to manipulate the figure, first by dragging and after by animation. The students observe the changes in the figure without describing the phenomenon. They also measure the sides and the area, by using one of the functions of Cabri. Measuring is a necessary stage toward the analytical representation of the situation, which is implied by the second question in the statement. We point out that there is no reflection on these numerical data, the students are only collecting all the available data.

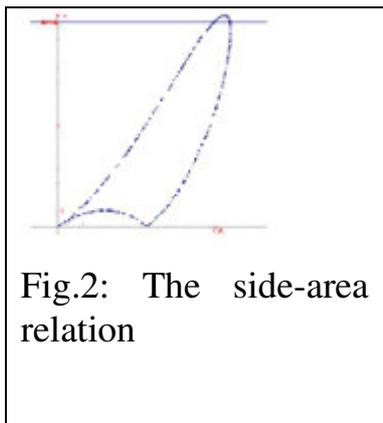
This is the phase in which the students "get in touch with the problem": they are reflecting, but not yet producing ideas. In this step the communication between mates is confined to the exchange of technical instructions ("move the mouse", "measure",...). Here Cabri seems to foster a permanence in the embodied mode of representation, see (Tall, 2003), with prevalence of the enactive mode.

Second episode: working on the analytical representation

Alessio proposes to produce an analytical representation. This is the crucial point, since this decision entails to choose two entities to represent on the axis. The first natural choice of variable is the area. The other variable may be the side BC , or the side AC , or the height of the triangle (which is not yet drawn on the figure and whose measure, consequently, is not on the screen). Andrea proposes to choose as variables

the area and the side AC . Alessio proposes to choose as variables the area and the sum of the two sides AC and BC . We guess that Alessio tries to take into account *as much data as possible*, namely *all* the things that vary in the figure: the area and the two sides (the third side is the fixed chord). This fact brings to the fore that for the moment the choice of variables comes from the figure, where the height is not present, and not from a more formal reflection on the area based on the formula $(\text{base} \cdot \text{height})/2$.

The group agrees to Andrea's proposal. At first, the students observe the point moving across the Cartesian plane. After, they produce the trace of the movement of the point and they start the actual reflection on the diagram (see fig.2).



Andrea observes that “it rises until it reaches a maximum value”. The students pay attention to particular points of the diagram (such as maximum and minimum) and link them to the geometrical configuration. We realise there are characters of the graph which are more evident and telling to the students, such as the existence of a maximum, respect to the trend of the diagram. This behaviour encompasses a ‘causal’ interpretation of the graph: when the point C is there, the point on the graph is...

We point out that the students’ mode is still enactive, since they do not go beyond what the physical action of dragging shows.

The successive step is to reflect on the presence of two pieces of curve (see fig. 2). The students work back and forth from the diagram to the geometric figure and they understand there are two pieces of graph because of the movement of the point C along two arcs of the circle. The figure is helping to understand the graph, that is there is interplay between the two ways of representing the situation in Cabri.

We also observe that they did not previously try to guess the shape of the diagram, by means of the study of the geometric phenomenon. This is in line with their conception of the graph in Cabri as an instrument to better explore the situation, rather than the representation of the result of an exploration. The approach to the problem is operational.

Third episode: producing and testing a conjecture

Alessio proposes his conjecture about the maximum value of area: he thinks the maximum area is obtained when the point C is in the intersection of the circle and the perpendicular bisector of the chord AB (that is to say the triangle is isosceles). This conjecture comes from the perception of the figure supported by the check on numerical values. He constructs the perpendicular bisector and drags the point C to the intersection in order to test his conjecture. The transcript shows that the conjecture is tested also through numerical values.

Observer: Where did you see that the area is the maximum?

Alessio: We saw it both in the graph and in the numerical values.

We have an example of dragging test modality, see (Arzarello et al., 2002), performed also with the numbers. We note that, once the dragging confirms the conjecture, the students don't feel the need of a formal justification, that is to say they don't look for the reason why the maximum area is reached in that configuration. They act operationally.

Forth episode: searching for a justification

The teacher gives some prompts addressed to enlarge the scope of the students' work. He suggests to reflect on the numbers that are on the screen and to explain why the conjecture holds, rather than just testing it.

We report a short excerpt from the transcription of the video:

Andrea: The teacher asked why the triangle isosceles is the triangle of maximum area.

Alessio: I was thinking that... since the area of the triangle is $b \cdot h / 2$ [he writes down on the paper the formula], if the base is fixed, the area depends from the height. In this case, if we have the maximum height we have the maximum area. [...] How can we say that it is the maximum height?

Nicolas: The triangle isosceles is the triangle of maximum area and the triangle of maximum height [among the triangles that may be inscribed in a circle].

We observe that the intervention of the teacher pushes the students to reflect on the reasons, which are behind their results. In this way they pass from the exploration on graph/figure to the reflection on the formula of area. This reflection brings to the fore a hidden element, which was not yet present on the figure: the height. As a consequence, Alessio proposes to do a height-area diagram. In this way they start a new exploration, with a new element that comes from theoretical reflection. We point out that this new element does not change their way of looking at the problem. This moment could have marked the passage from an embodied/operational way of looking at the problem to a symbolic/proceptual one, but this did not happen. They do not exploit the formula, but immediately they are *attracted* again to the exploration with Cabri. As we'll see in the fifth episode, the teacher intervenes with the aim of making the students to exploit the insight coming from the algebraic formula and to produce a more abstract thinking.

Fifth episode: Foreseeing the new diagram

The teacher encourages the group to describe the kind of diagram they expect to obtain.

Teacher: Before doing the graph, try to imagine how it can be.

Alessio: When the point C is next A or B , the height is shorter and then the area is smaller.

Teacher: OK, but do you remember the previous graph? The new graph will be the same?

The three students reflect on the figure and agree the diagram will be “the same as the previous one” (see Fig. 2). They are led to this conclusion by a rough reflection on the manipulation of the figure (compare Andrea’s sentence): they do not see any difference between the phenomenon of the area increasing or decreasing according to the variation of the height, and the phenomenon previously observed (area increasing or decreasing according to the variation of one side). Thus, they conclude the new diagram will be the same. Here we have that the structural view of the diagram is very poor; there is not the concept of the kind of variation, but only the naive idea of growing, diminishing, maximum, minimum. The students are still acting in the embodied mode.

The formula of area, which was reminded before, does not provide further elements of reflection. The students don’t reflect on the formula to anticipate the diagram. We may say in this case they lack of anticipatory thinking, but we are aware that the functions of Cabri, by making direct and immediate the result of transformations and decisions, may hinder the role of anticipatory thinking. What seems to us interesting is that, even when they are forced to anticipate by the teacher, they rely on figure (manipulation of the figure through Cabri) rather than reflecting on the formula.

Sixth episode: Introducing numbers

The teacher suggests to resort to numbers (measures) as a further information to study the nature of the graph. The aim of the teacher is to make the students aware of the difference between the two relationships. The key concept is the way of changing of numbers. But, as we’ll see in the following transcript, this peculiarity is not evident to the students. Andrea’s attention is caught mainly by the covariance.

Teacher: Let’s look also at the numbers, how are they varying?

Andrea: The rate of change of the numbers ... this means that if it [the height] increases or decreases more and more... here it [the area] increases more and more ... and decreases more and more.

Seventh episode: the surprising diagram

At this point the students have at disposal four poles of interest, that is four representations of the problem: geometric (the dynamic figure), analytic (the diagram), numerical (the measures) and algebraic (the formula of area). Up to now they have worked mainly between the geometric and analytic poles. The lack of meaningful interaction of the four poles is proved by the fact they are very surprised when Cabri gives the actual diagram, see fig. 3. Andrea claims “It is a

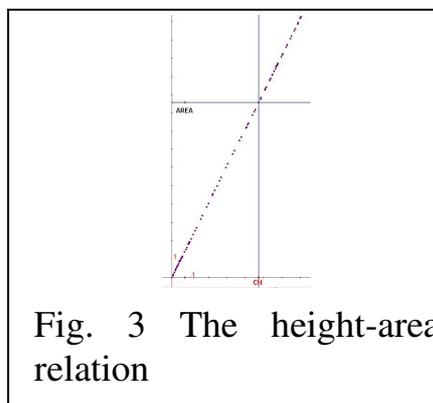


Fig. 3 The height-area relation

linear function”, Alessio looks at his notes where he wrote the formula of area (see the fourth episode), but nobody makes explicit that the linearity was already present in the formula.

Once more, we observe they don't rely on formula as a source of understanding.

Eighth episode: looking at numbers ... again

At this point, the teacher intervenes in order to foster the interplay between modes of representation. The intervention of the teacher leads the students to explain the surprising diagram they obtained: the diagram is quite different from the previous one, even if the geometric situation seemed analogous (variation of the area according to the variation of the side/height). He reminds that the students had also reflected on measures to check this conjecture, see the sixth episode.

Teacher: What did the numbers suggest?

[...]

Teacher: There seems to be a contradiction between numerical and graphical aspects. From the numbers we expected a different thing [graph]...

Andrea: At first I looked at the height, now I'm trying to look at the values of the two sides, in order to see if...because I think we should see something. We saw that the variation of the height and the area gives a graph, the variation of the numbers [the two sides] should be different. It should be different because the graph is different. [...] Here... the variation of the area follows the variation of the height. The area decreases and the height decreases, while for the sides... the area decreases and the side still increases. When the area varies, height and side vary in a different way. Area and height should vary in the same way: when the height increases the area increases, for the side it is not the same thing.

The numbers play a key role in changing the way of looking at the situation. The students grasp that in the covariance it is important not only what is varying, but also the way in which it varies. We see here the teacher's key role in scaffolding the students' understanding.

What seems to be still a source of problems is the 'formal' justification of the linear dependence. (of course, we do not focus on the lack of formal reflection on the non-linear situation of the relation side BC-area). The students have at disposal the formula and are aware (from the diagram) that it is a linear dependence. What they lack is a semantic control of the manipulation of the formula.

CONCLUSIONS

To answer our initial questions we summarise our comments to the episodes. The aim of the activity was to lead the students from the exploration of the problem to the construction of concepts (covariance, linear dependence). This path should have implied to pass from an operational view of the problem to a structural view based on the symbolic modes (numeric/algebraic and logic/verbal). Through Cabri four poles

of interest, that is four modes of representation, are available: geometric (the dynamic figure), analytic (the diagram), numerical (the measures) and algebraic (the formula of area). The students deal easily with geometric and analytic poles, but their mode of working is enactive/iconic so that we may say that their way of approaching the problem is operational. Numbers (here measures) are used spontaneously mainly to support dragging exploration, but the students need the help of the teacher to make numbers become source of insight for the problem.

Our experiment offers an instantiation of the way the path from the embodied stage to formal mathematics may happen in a rich dynamic geometric environment. The purpose of the exercise was not to reach the formal-axiomatic representational world; anyway, we note that even the symbolic-proceptual world is hindered to students: the dynamic figure and the diagram may act as a burden in the effort to go on.

From our experiment we may draw some didactical implications. Our exercise stresses all the potentialities of dynamic geometric environments. Without Cabri the students would have missed the exploration of the role of dependent and independent variables that emerged in our experiment and would have not had the deep insight into linear dependence; really through Cabri they had the chance to face all the complexity of the phenomenon. On the other hand, to deal with the richness of information turns out to be more complex and requires the flexibility for switching from one pole to another.

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