MOZAMBICAN TEACHERS' PROFESSIONAL KNOWLEDGE ABOUT LIMITS OF FUNCTIONS

Danielle Huillet

Eduardo Mondlane University

This paper analyses five Mozambican secondary school teachers' professional knowledge about limits of functions. Building up on Even's analysis of SMK, a new framework has been put together for this analysis. Interviews have been held with the teachers to analyse their knowledge according to this framework. Data suggest that the teachers have a very weak knowledge of the limit concept.

INTRODUCTION

This work is part of a research project that aims to investigate how high school mathematics teachers' knowledge of limits of functions evolves through their participation in a research group.

The limit concept has been chosen because it is the first higher Mathematics concept met by students in secondary schools. It is a very abstract concept and difficulties experienced by students have been explained as caused by a gap between the concept-definition and the concept-image (Tall & Vinner, 1981). Several studies in Mathematics Education about the concept of limits have already been done. In a European context, these studies relate to students conceptions (Tall & Vinner, 1981; Cornu, 1984; Sierpinska, 1987), epistemological issues (Cornu, 1991; Sierpinska, 1985; Schneider, 2001) or results of didactical engineering (Robinet, 1983; Trouche, 1996). In the Mozambican context, they relate to the institutional relation (Chevallard, 1992) of secondary education (Mutemba & Huillet, 1999), the personal relation of some teachers (Huillet & Mutemba, 2000), and students' understanding of limits (Mutemba, 2002). In my research, I focus on the professional knowledge of mathematics teachers on this concept.

What kind of knowledge does a teacher need to teach a mathematical topic? Several authors tried to answer this question. In particular, Even (1993) considers teachers' knowledge about a mathematical topic as having two main components: teachers' subject-matter knowledge (SMK) and pedagogical content knowledge (PCK). She states that few years ago, teachers' subject-matter knowledge was defined in quantitative terms but that,

> in recent years, teachers' subject-matter knowledge has been analysed and approached more qualitatively, emphasising knowledge and understanding of facts, concepts and principles and the ways in which they are organised, as well as knowledge about the discipline (p. 94).

Pedagogical-matter knowledge

> is described as knowing the ways of representing and formulating the subject matter that make comprehensible to others as well as understanding what makes the learning of specific topics easy or difficult (pp. 94-95).

She (Even, 1990) built an analytic framework of SMK for teaching a specific topic in Mathematics, applied to the study of the function concept, that considers seven main facets of this knowledge: Essential Features, Different Representations, Alternative Ways of Approaching, The Strength of the Concept, Basic Repertoire, Knowledge and Understanding of the Concept, and Knowledge about Mathematics.

Looking at teachers' pedagogical content knowledge of Geometry, Rossouw & Smith (1998) describe PCK as "a means to identify teaching expertise which is local, part of the teachers' personal knowledge and experience" including

> (a) the different ways of representing and formulating the subject matter to make it comprehensible to others, (b) understanding what makes the teaching of specific topics easy or difficult and (c) knowing the conceptions and pre-conceptions that learners bring to the learning situation (p. 57-58).

They also referred to Mark (1990), who

> has painted a portrait of PCK as composed of four major areas: (a) knowledge of subject matter, (b) knowledge of student understanding, (c) knowledge of the instructional process and (d) knowledge of the media for instruction (p. 58).

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Analysing these frameworks and applying them to the limit concept, I found that the boundary between SMK and PCK was not very strict, and that this distinction was not so relevant to my study. In order to teach limits, a teacher needs to have a deep knowledge and understanding of this topic, and this knowledge must be oriented towards teaching it to specific students in a specific context. Consequently, in my study of teachers' knowledge about limits of functions, I do not distinguish between SMK and PCK, but consider what kind of knowledge a teacher needs for teaching limits of functions in Mozambican secondary school. My framework includes some of the topics of Even's SMK, and the part related to students understanding, conceptions and difficulties, taken from Rossouw and Mark. In this way, the components of teachers' knowledge about limits of functions that I consider are: (a) Essential Features; (b) Different Settings and Models; (c) Different Ways of Introducing the Concept; (d) The Strength of the Concept; (e) Basic Repertoire; (f) Knowledge about Mathematics; (g) Students Conceptions and Difficulties. Obviously these seven aspects are not strictly separated, and some of them are strongly interrelated.

I used this framework for interviewing the teachers and for defining the topics for their personal research.

METHODOLOGY

At the beginning of the study, six teachers were selected (*A* to *F*), each one researching a specific aspect of the limit concept. One of them (*B*) dropped out at the very beginning of the work. The methodology for the whole research included three interviews with each teacher, individual supervision sessions, and periodic seminars where they discussed their personal research or a specific aspect of limits. The results presented here come from the first interview, held at the beginning of the whole process. It was my first individual contact with the teachers after they decided to join the group. In order to create a good relationship with them at this early phase of our work together, I did not want them to consider this interview as trying to test their knowledge about limits of functions, but as a conversation about this concept. For this reason I conducted semi-structured interviews focusing on the story of their contact with "limits of functions" through the several institutions where they had met this concept (Chevallard, 1992), as well as on their personal ideas about the teaching and learning of limits of functions at school. During the interview, the teachers were shown several definitions and several tasks about limits in different settings (numerical, graphical, and algebraic) and were asked which of these definitions or tasks they would use to teach in secondary school. They were not asked to solve the tasks, but a few of them voluntarily engaged in solving them.

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In this section, I will present some results of the analysis of the first interview, according to the seven aspects of my framework.

Essential Features

The essential features pay attention to the essence of the concept and deal with the concept image. I consider three main aspects of the limit concept, which emerge from the history of the limit concept and have already been underscored by Trouche (1996). These are:

- A dynamic or "cinematic" point of view, related to the idea of movement;
- A static or "approximation" point of view: the approximation of the variable depends on the degree of approximation needed;
- An operational point of view: the limit works in accordance with specific rules.

Through their answers to the question "*How would you explain the limit concept to a person who doesn't know Mathematics, for example a Portuguese teacher?*" and other statements during the interview, the teachers showed that they mainly consider limits as algebraic calculations. They also evidenced a static view of limits, as they described it as an unreachable approximation (*A*), a boundary (*C*), a repetition (*D*), a maximum or a minimum (*E*), and a limitation or restriction (*F*). Two of them also spoke about a dynamic point of view (*A* and *C*).

Different Settings and Models

Several authors pointed out that a mathematical object can be represented using different models (Chevallard, 1989) within different settings (Douady, 1986) or representations (Janvier, 1987). Changes of settings, or shifts from one model to another, allow the learner to access new information about the concept and, consequently, to construct a deeper understanding of this concept.

The limit concept can be studied in many different settings: geometrical (areas and volumes), numerical (sequences, decimals and real numbers, series), cinematic (instantaneous velocity and acceleration), functional (maximum and minimum problems), graphical (tangent line, asymptotes, sketching the graph of a function), formal (ε-δ definition), topological (topological definition, concept of neighbourhood), linguistic (link between natural and symbolic languages of limits), algebraic (limits calculations). Each of these settings underscores a specific feature of the limit concept.

In Mozambican secondary schools, the algebraic setting is dominant (Mutemba & Huillet, 1999). To analyse the teachers' knowledge about settings, I used different tasks in different settings, in particular numerical and graphical setting, which are unusual in Mozambique, and asked the teachers their opinion on the use of these tasks at secondary school. Some of them tried to solve the tasks.

The teachers showed that they were able to solve tasks in an algebraic setting.

C and *D* did not understand the tasks in a numerical setting. *A*, *E* and *F* recognised some numerical tasks but were not very used to solve them.

The graphical setting is where the teachers faced more difficulties. I showed them several tasks, some of them to read limits from graphs, and other to sketch a graph using limits, without analytical expressions. Three teachers (*a*, *e* and *f*) did not try to solve the tasks to read limits. *A* tried to relate all graphs to some analytical expression. *C* was able to solve most of these tasks, sometimes after some hesitation. *D* tried to solve some of the tasks but faced many difficulties and said that he had never done it before. *E* and *F* did not even try to solve any of the tasks.

Regarding the tasks on sketching graphs using limits, *A* only tried to solve one of them and faced many difficulties. *C* solved correctly one of them, showed interest for this kind of task, but did not try to solve the other ones. *D* tried to solve one of the tasks but sketched a graph that did not represent a function. *E* and *F* did not try to solve any of these tasks.

Their knowledge of the formal setting is also very weak (Huillet, 2004).

Alternative Ways of Introducing

There are several ways of introducing a concept at school. For instance, the limit concept can be introduced through sequences, through the tangent line problem, through problems of maximum or minimum, through instantaneous velocity or through the formal definition. These different ways of introduction correspond to different settings and underscore a specific feature of the limit concept.

During the interview, I asked the teachers the following questions:

> At secondary school in Mozambique, limits are usually introduced through sequences. What do you think about this way of approaching limits? What other ways of approaching limits do you think could be used at school? Which one do you think more appropriate to secondary school level?

All teachers know the way limits of functions are usually introduced in Mozambique, according to the syllabus. None of them presented any alternative to this method. Two of them spoke about using more graphs (*C* and *E*) but without explaining how to do it.

The Strength of the Concept

The strength of a concept deals with the importance and power of this concept, with what make this concept unique. In that sense the concept of limits of functions is a very strong concept. It has strong links with other mathematical concepts, such as the function and the infinity concepts. It is also a basic concept for differential and integral calculus. Furthermore, it has many applications in other disciplines, such as Physics, Biology and Economics.

To analyse this aspect, I asked the teachers the following questions:

> In your opinion, what kind of applications of the limit concept should be taught at school? Do you think that it is useful to teach limits of functions at secondary school level? Why? How do you think the students will use this concept later, during their studies at university for example? In which disciplines? In which areas?

The teachers pointed the following applications of limits: in physics (*A*, *E* and *F*), in geometry (*A*), in derivatives (*C* and *F*), to locate intervals of increase and decrease of a function (*D*) and in the convergence of a series (*F*). *C* said that he does not understand the importance of the limit concept and *D* that he does not see it as a special concept.

Basic Repertoire

According to Even (1990), the basic repertoire of a mathematical topic or concept

> includes powerful examples that illustrate important principles, properties, theorems, etc. Acquiring the basic repertoire gives insights into and a deeper understanding of general and more complicated knowledge" (p. 525).

In Mozambique two kinds of tasks are usually solved in secondary schools: those to calculate limits and those to study the continuity of a function (Mutemba & Huillet, 1999). During the interview, it became clear that, either through their own experience with the concept or through their opinion on the tasks presented to them, the basic repertoire of the teachers was limited to the kind of tasks usually solved in

Mozambican schools: algebraic tasks to calculate indeterminate forms, and some tasks to apply the ε-δ definition.

Students Conceptions and Difficulties

When teaching a mathematical topic, it is important that the teacher is aware of the different conceptions, and even "misconceptions" or "alternative conceptions", held by the students, as well as the difficulties they face. Some of the conceptions and difficulties of the students when learning the limit concept have already been highlighted by several researchers (Cornu, 1983; 1991; Sierpinska, 1985; 1987; Monaghan, 1991).

In the interviews, I asked the questions:

> Which difficulties do you think that students meet when they study the limit concept? How do you explain these difficulties?

In answering these questions, the two teachers (A and E) who already taught limits at school used their experience as teachers and the others (C, D and F) their own experience as students. They pointed out students' difficulties to understand the ε-δ definition (A, C, E), specifically because of the use of Greek letters (E), to use certain techniques to calculate some indeterminate limits (E) and to read graphs (F).

Knowledge about Mathematics

In this section, I analysed what kind of knowledge about Mathematics can be helpful to learn limits of functions and, at the same time, how teachers' knowledge about mathematics can be developed through the study of this topic. One important aspect, through the study of the ε-δ definition, would be reflecting on the role of definitions in mathematics. It would also help them to reflect on the role of proofs: why is it necessary to prove that the limit is *b*, using the definition, if we already calculated the limit and found *b*? Another important aspect is the connectedness, as stated by Ball et al. (2004):

> Another important aspect of knowledge for teaching is its **connectedness**, both across mathematical domains at a given level, and across time as mathematical ideas develop and extend. Teaching requires teachers to help students connect ideas they are learning. […] Teaching involves making connections across mathematical domains, helping students build links and coherence in their knowledge (p. 59-60).

The limit concept has different features, can be studied in several settings and has strong links with other mathematical concepts. This should help the students to build links and coherence in their knowledge.

I did not ask specific questions about this aspect, but from the teachers' discourse it was clear to me that their knowledge about mathematics is very weak. They are used to learn rules without demonstrations and they are not able to make the connection between different concepts or between different settings.

CONCLUSION

Summarising the main results of the interviews according to my framework on teachers' professional knowledge about limits of functions, I would say that:

- They mainly consider limits according to operational and static points of view;
- They are used to work with limits in an algebraic setting, and face many difficulties in linking it with a graphical setting; they also face some difficulties when working in numerical and formal settings;
- They only know the way of introducing limits that is stated by the Mozambican syllabus;
- They do not understand the strength of this concept, as they know very few applications in mathematics and in other sciences;
- Their basic repertoire is limited to algebraic tasks and some tasks with the ε-δ definition;
- Their knowledge of students conceptions and difficulties is limited by their own knowledge about limits;
- Their knowledge about mathematics is also very weak.

As a conclusion, I would say that these teachers showed a weak knowledge of the limit concept, mainly shaped by the institutional relation of Mozambican secondary school to this concept.

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