

A PROCESS OF ABSTRACTION BY REPRESENTATIONS OF CONCEPTS

N.C.Verhoef & H.G.B. Broekman

University of Twente, Utrecht University

The purpose of this article is to describe the integration of epistemological principles, theories about levels of argumentation, and different worlds of abstraction to address secondary school students' development of mathematical concepts. A pilot study concerning 'volume and enlargements' focuses on step-by-step solutions of classified problems to establish the progress of the process of abstraction. The analysis is based on different language use.

INTRODUCTION

Worldwide, but especially in the Netherlands, mathematical education illuminates only a few characteristics of a preparation for a scientific study at a university. In addition mathematical concepts are wrapped in a variety of contexts. Subsequently it is necessary to begin a process of abstraction by releasing these contexts. Otherwise students' intellectual challenge decreases more and more. Casually the attention of (supposed) applications and applicable meanings dominate mathematical education too long. The development of a process of thinking unfastened of applications starts abruptly, without systematically foundation. Dutch textbooks demonstrate a mostly absence of definitions and development of a theory, and an elevation of illustrations and visualizations to mathematical concepts. Students and teachers alienate from their intuition by acquired tricks and bad methods of learning and teaching. Mathematical education, answering this characterization, lacks the intellectual challenge highlighted by a scientifically study in the future.

THEORETICAL FRAMEWORK

Teaching and learning with representations of objects

From a cognitive psychological point of view the development of knowledge takes place by experiencing reality (Piaget, 1972). Experience results from the interaction with objects which on the one hand leads to the development of ideas about these objects and on the other hand by manipulation to the possibility to evaluate whether these ideas are correct or not. Experiences with objects in the real world can be divided into (a) direct experiences with the reality or (b) mediated experiences by the use of media. In the latter hand a medium is used to depict or to describe the reality or both. In education, mediated experience by using a representation of the reality is essential, as real objects are not always available or suitable to use. Besides it is possible to refer to an imaging reality.

The representing medium (the representation) is related to the represented object (the reality) through a set of mapping principles that maps elements of the reality to elements in the representation. Some representations are (almost) similar to the represented object, such as photographs or statues. These are called *pictures*. In these cases, every element in the represented object is represented by a unique element in the representing medium, so that there is a one-to-one mapping or isomorphism between the two. If a representation is an abstraction of the represented object in which the characteristics relevant to the situation emerge and in which the characteristics irrelevant to the situation are left out, this is called homomorphism between the representing medium and the represented object. In this case, two or more elements in the represented object are represented by only one element in the representing medium. An example of this is the figure of a man or a woman on a toilet door. In all cases in which a representation represents the represented object to some extent of similarity, the term *icon* is used. The relationship between an icon and the represented object depends on their ‘mode of correspondence’. There are also representations that have no similarity at all with their represented object. These are chosen arbitrarily by convention and are called *symbols*. Examples of these are the letters of the alphabet, or numerals.

From instructional design basic ideas Seel and Winn (1997, p. 298) argued that ‘people’s thinking consists of the use and manipulation of signs as media for the representations of ideas as well as objects’. Changes in the kind of representations will have a direct effect on learning processes. In this study about the development of mathematical concepts the reference point is the use of signs as: ‘pictures’ (a video or a photograph); ‘icons’ (a figure); and ‘symbols’ (a formula or a definition).

Teaching and learning mathematics by reasoning and argumentation

In mathematics there are frequently used signs at different levels of abstraction. For instance the representation of the concept ‘circle’ is a photograph of a cup of tea (picture), a round about on paper (icon), or a set of points (symbol).

Particularly the Dutch researcher Pierre van Hiele (1986) was engaged in levels of mathematical thinking, especially in geometry. He introduced levels of reasoning or argumentation to indicate a process of abstraction: the zero-level (visual level) of sensorial perceiving of objects, the first level (description level) of properties of objects, and the second level (theoretical level) of sets, logical operators and formal proofs. There exist some variants with another third level. In that case the second level is called the informal deductive level and the third level is mentioned the deductive level. Later on this variant comes back in this article. Van Hiele characterised the zero-level, or called the ground level, as a lack of relationships. The basic ideas about mathematical concepts rest on intuition. Either at the first level the concepts are founded on the properties of mathematical concepts. At this level it is possible to manipulate concepts because of the understanding properties. By Van Hiele the transition from the zero-level to the first level exists of the determination of

sensorial descriptions. So relationships will be built as a net of relations based on knots of mathematical concepts with a pack of properties each. In accordance with Van Hiele it is possible to test the presence of a net of relations by the determination of the usage of language. Van Hiele supposed at the transition of the zero-level to the first level a developmental process of language use from everyday language to formal language. For example the judgment “the direction NNW (bisector) at this map lies exactly between the direction N and the direction NW” in everyday language and “each point at the bisector has the same distances to both sides of the angle” in formal language. The second level of Van Hiele distinguishes a total disconnection of real situations or schemes. The properties of mathematical concepts are logical ordered by which arises a formal relationship between the concepts. At this level it is possible to deduce formal properties from other properties. So proofs as well as the usability of symbols, definitions and formulas are necessary to describe mathematical concepts. By Van Hiele the transition from the first to the second level occurs in a process of analysis and objectivity by a development in language use from formal to symbol language.

Teaching and learning mathematics by a journey through 3 worlds

Also David Tall (2004) is intensively engaged in all about abstraction in mathematics education. He distinguishes explicitly the development of geometric and algebraic mathematical concepts. So he presumes three different worlds: the ‘embodied world’, the ‘symbolic world’, and the ‘formal world’.

In the first embodied world the development of concepts is realized by a growing process that starts in the real world and consists of our thinking about things that will be perceived and sensed, not only in the physical world, but also in an individual mental world of meaning. By reflection and by the use of increasingly sophisticated language, it is possible to focus on aspects of sensory experiences that enables to envisage conceptions that no longer exist in the real world outside, such as a ‘point’ that has no thickness, no length and no width. This world mentions the ‘conceptual-embodied world’ or ‘embodied world’ for short (Tall & Ramos, 2004). This includes not only mental perceptions of real-world objects, but also intentional conceptions that involve visuo-spatial imaginary. It applies not only the conceptual development of geometrical objects but also other mathematical concepts like algebraically (square root of number two), analytical (derivative) and statistical objects (median).

The second world is the world of symbols that are necessary for calculation and for manipulation in arithmetic, algebra, calculus, statistics and so on. These begin with actions that are encapsulated as concepts by using symbols that allow an effortless switch from processes to do mathematics to concepts to think about. This second world is called ‘proceptual (the term procept is a contraction of the terms process and concept) - symbolic world’ or ‘symbolic world’ for short. In this world the central point of view consists of actions with objects.

The third world is the formal world based on properties, expressed in terms of formal definitions that are used as axioms to specify mathematical structures (such as ‘group’, ‘field’, ‘set’). This third world is termed the ‘formal-axiomatic world’ or ‘formal world’ for short. This world considers of mental activities. Properties are used to define mathematical structures in terms of special properties. New concepts can be defined to build a coherent logically deduced theory.

Individual travellers through these three worlds take a unique route. Various obstacles occur on the way that requires earlier ideas to be reconsidered and reconstructed, so that the journey is not the same for each traveller. Either to attain the formal world it is recommended to start experiences in the embodied world, and to continue these experiences in the symbol world. Because of the development of geometrical concepts the journey to the formal world concerns only the embodied world by experiences and thought- experiences. This journey follows a natural growing process of sophistication via four Van Hiele levels of abstraction. The first step is the step of perceiving objects as whole gestalts. The second step is the step of description of properties with language growing more sophisticated. In a way that descriptions in the third step become definitions suitable for process and proof in the fourth step to the formal world. This journey provides a theoretical framework for the developmental process of learning geometry.

THE PILOT STUDY TO DESCRIBE CLASSROOM PRACTICES

In this pilot study the epistemological ideas (Seel & Winn, 1997), theories about levels of argumentation (Van Hiele, 1986) and different worlds of abstraction (Tall, 2004) are combined with each other to design principles of teaching and learning mathematics. For instance the step-by-step developmental process of the geometrical concept ‘area of a triangle’ can be activated by the usage of signs in Tall’s embodied world. That means assignments with representations of this concept at different levels of abstraction. A transparent paper with squares covers a photograph of a flat triangular object. The number of the squares is a measure of the area. Little squares result in another measure than big squares do. At this level the representations are based on sensory perceptions. The arguments are formulated in everyday language use. This is the first step in the developmental process to attain abstraction. The animated joining of the triangle and the encircled rectangle supply a foresight (properties of) the relationship between the area of the triangle and the area of the encircled rectangle. The language use change over to formal language. This is the second step. Afterwards the representation by a drawing of a triangle with base and altitude is enough to understand this concept. The language use reconverts to symbol language use. This is the third step. At the highest level of abstraction even the drawing is not necessary. The representation of the area of a triangle by the formula is enough. This is the last step in the process of abstraction. Finally the representation of the area of a triangle consists of a relationship between variables in Tall’s formal world. This is strictly mental and therefore maximal manoeuvrable. In terms of Van

Hiele the argumentation in everyday language use is founded on representations (signs) at the zero-level like photos (pictures), at the first level in formal language use like figures (icons), and at the second level in symbol language use like formulas (symbols).

Classified problems to structure the concept developmental process

The concept developmental process would be activated and stimulated by problems related to each level of abstraction, or each representation (picture – icon – symbol).

To structure the process of concept development the problems are classified in three different types: (1) categorisation problems, (2) declaration problems, and (3) design problems (Van Merriënboer & Dijkstra, 1997). In case of categorisation problems, mathematical objects must be assigned to unknown real or imagined situations. The strategy to solve categorisation problems underpins the use of a variety of visual and dynamic representations of mathematical objects.

In case of declaration problems, the cognitive constructs or declarative knowledge are principles, as well as casual networks and explanatory theories. The strategy to solve declaration problems is based on the declaration of relationships between mathematical concepts, manipulations with mathematical concepts, and applications with mathematical concepts. In this strategy, the focus is based on students' predictions concerning what will happen in specified situations, test predictions indicating whether it is confirmed or falsified, and, if relevant, specify the range of probabilities of occurrences of certain events. Explanatory theories predict changes of objects and relationships and lead to understanding of the casual mechanics involved. In the case of design problems, an artefact must be imagined and a plan has to be constructed to solve arising mathematical problems in the real world. The strategy to solve design problems concerns the construction of a model and the interpretations of models, or more models if necessary. For simplifying the reality and constructing a cognitive mathematical content, the label *mathematical model* is used. The mathematical model of the reality needs mathematical techniques (e.g. computer simulations) to be solved. Statements about the mathematical model will be retranslated in the modelling reality.

Step by step solutions to establish any progress in the developmental process

Problem solving skills are related to language use. To establish any progress in the concept developmental process the web-based problems were supplied to the - 12 or 13 years old - students with empty formats to preserve digital answers. The format was, based on principles of problem solving skills in heuristic mathematics education, divided into five steps (Van Streun, 1989): (1) application of the concept in the reality, (2) to order data, (3) to describe the approach, (4) to execute the approach, and (5) to reflect at the solution and at the approach. So students' notes could be sampled and compared step by step with everyday language use, formal language use, and symbol language use. For instance the following notes to assignment 14, paragraph (9.2):

Calculate the volume of a soup cup with the diameter 11,4 cm and the altitude 5,23 cm. Complete the answer at one decimal.

(1) volume of a circle, (2) diameter (11,4 cm) and altitude (5,23 cm), (3) to decide the volume of a cylinder by using the area of the ground plane (circle with radius 5,7 cm) and altitude (5,23 cm), (4) volume (soup cup) = $(\pi * 5,7^2) * 5,23 \approx 530,8 \text{ cm}^3$, (5) $530,8 \text{ cm}^3$ is a little bit more than half a litre can (reflection on the solution), the volume of a cylinder is like the area circle * altitude (reflection on the approach).

The pilot study to students' concept developmental processes

Twelve groups of secondary school students – 12 or 13 years old- of about thirty students each were involved in a three-week period where they applied a website with problems. All the problems were classified as categorization problems. The dynamical geometrical environment Cinderella functioned as a workspace. The subject was all about 'volumes and enlargements'. The paragraph was divided into sub-paragraphs: units of volume, volume of prisms and volume of cylinders, volume of pyramids, to enlarge and to reduce objects, area at enlargements, volume at enlargements, summary, mixed assignments, repetition, and extra, more creative assignments. All assignments were copied from an original, most common textbook, without any change. A picture illustrated each assignment. Ten participated teachers had the following instructions to support students' concept developmental learning processes: (i) to switch zigzag with students to representations of concepts at different levels of abstraction, and (ii) to search for other representations of concepts at the same level of abstraction (Verhoef, 2003). The data consisted of: (i) electronic students' step-by-step responses in the last (third) college time in the first week (pre-test) and in the ended (twelfth) college time in the last week (posttest), and (ii) a written test, the same test and the same teacher as the previous year. The electronic responses were transcribed and categorised by two student assistants into catchwords in everyday language, formal language and symbol language. The data were analysed at the language use. The expectation was (i) less everyday language use in the posttest in relation to the pre-test, and (ii) higher results in the written test than the results at the same test in the previous year.

The findings of this pilot study are related to epistemological principles as well as theories about teaching and learning in mathematics education.

In the pre-test students were inclined to annotate all the five steps. The possibility to answer step-by-step was a new phenomenon for these students. The solution (step 4) was in the post-test correctly annotated in formal language use and symbol language use. Students didn't use everyday language anymore.

The results of the written test were higher than the test results of previous year in three groups. Teachers of these groups required only students' step-by-step solutions. In the other groups the test results were averagely lesser. The low results were attributed to the incorrect use of symbol language use (formulas) without formal language use (descriptions).

DISCUSSION

This pilot study involved analysis of secondary school students' development of mathematical concepts by categorization problems and the possibility to answer step-by-step. Firstly it is necessary to investigate types of different problems like categorization problems, but also declaration and design problems. Each problem type has original skills to solve these problems, prepared on different representations (picture – icon – symbol), different language use (everyday – formal- symbol), and different (thought-) experiments.

Secondly, it seems advisable to answer step-by-step, because of higher results in written tests. Reflection (step 5) will be the most important step to attain a higher level of abstraction. Naturally students follow their intuition, they don't reflect (De Bock, Van Dooren, Verschaffel, & Janssens, 2002). So teacher's support focuses on the emphasis at the process of reflection highlighted by the use of representations.

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