

CHARACTERIZING MIDDLE SCHOOL STUDENTS' THINKING IN ESTIMATION

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The goal of the research reported was to develop a framework for describing students' thinking in situations involving computational estimation. Case-study methodology was used to investigate 8th grade students' abilities to estimate. The developed Estimation Thinking Framework is empirically based, and consists of four developmental levels: (1) predimensional, (2) unidimensional, (3) bidimensional, and (4) integrated bidimensional. These levels are hierarchical in nature and based on the data collected during the study, analysis of existing classifications of estimation strategies, as well as Case's (1996) theory of cognitive development.

Mathematics education reforms of the recent decades in the United States have recommended that school mathematics curricula include topics on estimation. The National Council of Teachers of Mathematics (NCTM), in its Principles and Standards for School Mathematics (NCTM, 2000), acknowledged the importance of developing students' ability to estimate. The attention given by NCTM to the necessity of developing estimation skills has resulted in the expansion of research on how students develop computational estimation strategies and how they reason in problem situations in which the context calls for an estimate.

Over the last three decades, a substantial volume of research has been accumulated and provides a theoretical foundation for understanding the development of students' abilities to estimate. However, further investigation of learning, understanding, and teaching estimation is necessary in order to provide educators with insights into practical applications of this research knowledge. The goal of the current study was to develop a framework for characterizing levels of students' thinking based on their choice of computational estimation strategies. The research, stemming from the analysis of existing classifications of estimation strategies (e.g., Reys, Rybolt, Bestgen, & Wyatt, 1982; Rubenstein, 1985; Levine, 1982) and Case's (1996) theory of cognitive development, addressed the construction of an empirically based framework which can be used to identify a level of middle school students' thinking with regard to estimation. Thus, the findings of the study shed light on why a student chooses a particular estimation strategy.

REVIEW OF RELATED LITERATURE

There seems to be a consensus among mathematics educators and researchers that instructional decisions should be grounded in research-based knowledge of student thinking (e.g., Carpenter & Fennema, 1989, 1993; Mack, 1995; Lamon, 1993). Carpenter et al. (1989) and Fennema et al. (1993) posited the need for cognitive frameworks that would provide researchers and educators with detailed knowledge

about children's thinking in each knowledge domain. This approach has been adopted by several studies targeting various mathematical knowledge domains. Carpenter et al. (1989) used this approach in a study focused on whole number arithmetic; Mack (1995) applied the same approach in the domain of fractions; Lamon (1993) utilized this principle in the domain of proportional reasoning; and Fuys, Geddes, and Tischler (1988) have proven this approach to be effective in the geometry domain. While the body of knowledge on development of children's ability to estimate is growing, there is still a need for a framework that will allow characterizing levels of students' thinking with regard to estimation.

Several researchers devoted their studies to identification and classification of estimation strategies. Rubenstein (1985) examined the computational estimation abilities of eighth-graders, and developed an estimation test to measure four types of computational estimation (open-ended, reasonable versus unreasonable, reference number, and order of magnitude). Whereas Rubenstein (1985) classified estimation strategies based on the type of problems the students were given, Levine (1982) suggested a model for classification of estimation strategies based on the type of estimation technique. The model consisted of 8 categories: (1) using fractions; (2) using exponents; (3) rounding both numbers; (4) rounding one number; (5) using powers of 10; (6) using "known"/ "nicer" numbers; (7) incomplete partial products; and (8) proceeding algorithmically.

Levine's (1982) model is limited in its application: it can be used for strategy identification purposes only, and provides no way to account for all the possible estimation strategies. However, the nature of computational estimation is such that the number of estimation strategies is limited only by one's level of cognitive development. Dowker (1992) conducted a study on estimation strategies used by professional mathematicians. The results showed that mathematicians used a great variety of strategies, as many as 23 for a single problem. This presented a challenge for the researcher, who tried to encompass all the possible estimation strategies in one comprehensive classification. In the attempt to overcome such shortcomings, Dowker (1992) modified Levine's (1982) model by changing several categories and including the catch-all category named "other." This category included those strategies that were used only for one or two problems.

Reys et al. (1982) focused on "good estimators" in order to identify cognitive processes they use to solve problems that require computational estimation. The researchers administered a 55-item computational estimation test to over 1,200 students in grades 7–12 and to selected adults in order to identify a group of 59 "good" estimators. The people from the selected group were interviewed to determine strategies and processes they use to solve estimation problems. The three key cognitive processes people use in computational estimation were identified as: reformulation, translation, and compensation.

It is noteworthy that these classifications and models listed above, while providing some insight into the variety of estimation strategies and cognitive processes that

students use, do not focus on the students' progression through the levels of thinking in computational estimation. The current study developed a framework that can be used to characterize students' thinking with regard to computational estimation. The framework provides mathematics educators and teachers with descriptions of the levels of students' thinking with regard to computational estimation.

In 1996, Case put forth a theory of cognitive development with regard to quantitative thought. According to this theory, children's cognitive growth proceeds through two stages during the school years: the dimensional stage (approximate ages 5-10) and the vectorial stage (approximate ages 11-18). More importantly, Case (1996) identified the four sub-stages of each stage of the cognitive growth: (1) predimensional, (2) unidimensional, (3) bidimensional, and (4) integrated bidimensional. Based on the Case (1996) model of cognitive development the Estimation Thinking Framework was developed for describing and characterizing the development of student's thinking in computational estimation.

METHODOLOGY

Participants

For the current research, a case-study methodology was selected to construct descriptors of the developmental levels of the framework. Eight students in grade eight at a public school in Normal, Illinois, formed the population for the study. All of the students were chosen from the top level of their mathematics classes, on the assumption that the high-level achievement students could be expected to exhibit a greater variety of strategies and techniques (Reys et al., 1982; Dowker, 1992) in solving problems involving computational estimation.

Data Collection

This study utilized an original interview protocol, comprised of 19 tasks designed to assess students' ability to estimate across the four constructs: whole number, fractions, percent, and decimal fractions. The tasks allowed children to respond orally; students were not allowed to use paper and pencil to solve problems. The Interview Protocol was administered individually to each student in one 40-50 minute session; interview sessions were audio taped and transcribed. Students' responses to the Protocol tasks, along with the researcher's field notes, provided a basis for evaluating their levels of thinking in computational estimation.

Data Analysis

Transcripts of audiotapes and researcher's field notes on each student's thinking comprised the data for this study. Data were entered into a meta-matrix containing all students' responses to each question. A double coding procedure (Miles & Huberman, 1994) was used to code students' responses. First, the data were labeled, partitioned and clustered into the four categories (see Figure 1) according to the level of complexity of their estimation strategies: Level 1 (Predimensional), Level 2 (Unidimensional), Level 3 (Bidimensional), or Level 4 (Integrated Bidimensional).

Level	Descriptors of levels of cognitive development
Level 1 Predimensional	<p>Students who operate at this level:</p> <ul style="list-style-type: none"> • have a propensity to compute an exact answer, applying written or mental computation procedures and algorithms. • demonstrate inability to compare numbers using “benchmarks” (to identify which of two numbers is closer to a third number), to order numbers, to find or identify numbers between two given numbers • possess a limited understanding of the relations between numbers (e.g., multipliers vs. product, addends vs. sum, etc.) in arithmetic sentences.
Level 2 Unidimensional	<p>Students who operate at this level:</p> <ul style="list-style-type: none"> • use in addition to standard algorithmic procedures (mental or paper-and-pencil) other estimation strategies. For instance, the students might adapt and use rounding or front-end techniques with whole numbers. In problems involved fractions they might accommodate techniques that require converting fraction to decimal fractions. • tend to treat decimal fractions as whole numbers, employing rounding and truncating techniques. • might see the relationship between percents and decimals; however, tend to fall short of seeing connections between percents and fractions. • are limited in their use of the strategies, applying the same successful strategy over and over again for different types of problems.
Level 3 Bidimensional	<p>In contrast with students who function at Level 2 (unidimensional), students who operate at Level 3 (bidimensional):</p> <ul style="list-style-type: none"> • tend to use a greater variety of estimation strategies. • choose computational strategies in accordance with the situation described in the problem. • do not rely solely on a successful strategy that worked for other problems; become more flexible in their choice of strategies. • However, in contrast with students who function at Level 4 (integrated bidimensional), students who operate at Level 3 (bidimensional) • tend to use the estimation strategies that are available for them in isolation for each particular situation. • cannot easily switch from one strategy to another, which is illustrated by their inability to come up with different strategies for a single problem. • do not look for different possible techniques or strategies to verify their estimate
Level 4 Integrated bidimensional	<p>Students who operate at the integrated bidimensional level</p> <ul style="list-style-type: none"> • are able to coordinate two complex and multi-dimensional components (mental computation and number nearness task), • show their ability to switch between strategies easily. • are no longer applying the same strategy to “check” their estimates; they use different strategies to confirm the results. Moreover, once they find an estimate, they tend to try another strategy to yield a closer estimate.

Figure 1. Estimation Thinking Framework.

Next, students' responses to items within each category were described to yield criteria for each level of the initial framework. To assure the quality of the study, data were collected from two sources: (1) students' responses to the interview protocol tasks; and (2) researcher's field notes on each student's thinking strategies; the data from both sources were later tested for consistency.

RESULTS

The following subsections provide a parallel comparison of the descriptors of the Estimation Thinking Framework based on the data collected in the current study to the descriptors of sub-stages of the Case (1996) theory of cognitive development with regard to quantitative thought.

Level 1 – Predimensional

With regard to quantitative thought, Case (1996) found that the students who operated at the predimensional level were able to generate number tags (e.g., 2, 3, etc.) and to make qualitative judgments about quantities (e.g. more or less). However, they fell short of integrating these two aspects of knowledge into a meaningful structure, and therefore tended to respond at chance level when asked to decide whether, say, 4 was larger or smaller than 5. Results of the current study showed that students who operate at this level tend to use estimation strategies that are limited to standard algorithmic procedures. They have a propensity to compute an exact answer, applying written or mental computation procedures. They also show inability to compare numbers using "benchmarks" (to identify which of two numbers is closer to a third number), to order numbers, to find or identify numbers between two given numbers; as well as limited understanding of the relations between numbers (e.g., multipliers vs. product, addends vs. sum, etc.) in arithmetic sentences. For example, they are unable to predict what happens to the product if a multiplier is less than one, or if the multiplier is greater than one. Furthermore, when asked to estimate, students tend to respond at chance level by guessing the estimate.

The following is an excerpt from the interview with Masha (pseudonym), whose responses exemplify Level 1, "predimensional," of computational estimation.

Interviewer: Estimate the answer to " $3 \frac{1}{8} + 2 \frac{4}{5}$."

Masha: I'll start by converting this to improper fractions, so that'll be $\frac{25}{8}$ plus...

Interviewer: Do you think it's easier to work with improper fractions?

Masha: Yeah.

Interviewer: Can you just add these numbers, 3 and 2?..

Masha: Yeah. So, that would be 5 and then find the common denominator... I think it would be 40.

Interviewer: You don't need to find a common denominator, but think about fractions... $\frac{4}{5}$ and $\frac{1}{8}$. If you add these two fractions, is the sum greater or less than 1?

Masha: Greater...

It is clear from the above responses that Masha cannot think of any estimation strategies besides standard algorithmic procedure. Moreover, her predictions can be

described as a “lucky guess” strategy, since her answer is incorrect and she could not provide an explanation of her reasoning.

Level 2 – Unidimensional

According to Case (1996), students who exemplified thinking at unidimensional level were able to use a mental counting line to count backwards and forwards. However, they were unable to use one number line to locate two numbers and then compute differences simultaneously using the second number line. With regard to estimation, the results of the current study showed that, in addition to standard algorithmic procedures (mental or paper-and-pencil), students are able to use other estimation strategies. For instance, the students might adapt and use rounding or front-end techniques with whole numbers. In problems that involve fractions they might accommodate techniques that require converting fractions to decimal fractions. These students usually treat decimal fractions as whole numbers, employing rounding and truncating techniques. Even though the students operating at this level might see the relationship between percents and decimals, they fall short of seeing connections between percents and fractions. Despite the growing number of strategies available to students at the bidimensional level, they are still limited in their use of the strategies, applying the same successful strategy over and over again to different types of problems.

Level 3 – Bidimensional

Students who operated at Case’s bidimensional level showed proficiency at performing more than one mental operation, which Case calls vectors. Results of the current study showed that, in contrast with students who function at Level 2 (unidimensional), students who operate at Level 3 (bidimensional) tend to use a greater variety of estimation strategies. They choose computational strategies in accordance with the situation described in the problem. In other words, they do not rely solely on a successful strategy that worked for other problems; they become more flexible in their choice of strategies. However, the strategies are not connected and are used in isolation for each particular situation. The students cannot easily switch from one strategy to another, which is illustrated by their inability to come up with different strategies for a single problem. When they get an estimate, they stop looking for other possible techniques or strategies to verify their result.

Level 4 – Integrated Bidimensional

With regard to quantitative thought, Case (1996) found that the students who operated at the integrated bidimensional level were able to use multiple mental counting lines to do whole number arithmetic. With regard to computational estimation, the results of the study suggest that students at the integrated bidimensional level are not only able to coordinate two complex and multi-dimensional components (mental computation and number nearness task), but to do this they apply more than one strategy for a single problem. Thus, they show their ability to switch between strategies easily. At this level they are no longer applying

the same strategy to “check” their estimates; they use different strategies to confirm the results. Moreover, once they find an estimate, they tend to try another strategy to yield a closer estimate.

The following is an excerpt from the interview with Victor, who demonstrated his ability to apply two different strategies to find an estimate for the problem.

Interviewer: Estimate the answer to “ $3 \frac{1}{8} + 2 \frac{4}{5}$.”

Victor: I think five and something... a little less than six.

Interviewer: Why is that?

Victor: Because, uhm... these two [$\frac{1}{8}$ and $\frac{4}{5}$] don't look like they are enough to create one, so it would be more than five, because three and two make five...

Interviewer: But how do you know that $\frac{1}{8}$ and $\frac{4}{5}$ will not be enough to make a whole?

Victor: Just because one-fifths is greater than one-eighths.

Interviewer: Can you think of another way to justify that?

Victor: Uhm, because, $\frac{4}{5}$ is, like, 80% of 1.

Interviewer: Yes...

Victor: and then $\frac{1}{8}$ is, like, uhm... about 10% of 1.

Interviewer: Uh-huh...

Victor: So these two together are about 90%... So the answer will be close to 6...

It is clear from Victor's responses that, first, he applied a “benchmark” strategy – he estimated how far the two numbers are from the number 1, and how much it takes to complete another number to get the whole unit. Then Victor estimated what percent of the whole unit the fractions represent, and by adding two percents found that the fractions are not enough to make a whole. Thus, Victor's responses exemplify Level 4 thinking in computational estimation.

DISCUSSION

Data collected for the current research showed that children of the same age (in this case – middle school children) can be on different levels of thinking with regard to computational estimation. This finding is more in line with the van Hiele (1959/1984) model than with the Case (1996) model. While Case focused on students' age or maturation, the van Hiele model – although it does not specifically deal with the domain of computational estimation – places the emphasis on students' progression through the levels of understanding due to the instructional intervention. The levels of estimation thinking framework are not age-specific, and progress from level to level will depend more on the content and methods of instruction received by the student, than on their age. The presented framework was developed to characterize middle school students' thinking in estimation; however, further research is necessary to explore whether the framework is applicable to other age groups.

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